Quantum Machines

Les Houches July 2011

SQUID Amplifiers I [DRAFT]

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Contents

1. Introduction and overview

Superconducting QUantum Interference Devices (SQUIDs) are exceedingly sensitive detectors of magnetic flux. They are amazingly versatile, and are able to measure any physical quantity that can be converted to a flux, for example, magnetic field, magnetic field gradient, current, voltage, displacement, magnetic susceptibility, far infrared radiation, the density of axions (if they exist) and the state of a superconducting qubit. As a result, the applications of SQUIDs are wide ranging, from the detection of tiny magnetic fields produced by the human brain and the measurement of fluctuating geomagnetic fields in remote areas to the detection of gravity waves and the observation of spin noise in an ensemble of magnetic nuclei. Hundreds of thousands of SQUIDs are in operation today.

SQUIDs combine two physical phenomena, flux quantization, the fact that the flux Φ in a closed superconducting loop is quantized [1] in units of the flux quantum $\Phi_0 \equiv h/2e \approx 2.07 \times 10^{-15}$ Wb, and Josephson tunneling [2]. There are two kinds of SQUIDs. The first [3], the dc SQUID, consists of two Josephson junctions connected in parallel in a superconducting loop, and is so named because it can be operated with a static current bias. The second [4,5], the rf SQUID, involves a single Josephson junction interrupting the current flow around a superconducting loop, and is operated with a radiofrequency flux bias. In both cases, the output from the SQUID is periodic with period Φ_0 in the magnetic flux applied to the loop. Typically, above a few hertz, the flux noise is of the order of $10^{-6}\Phi_0\mathrm{Hz}^{-1/2}$, although for some devices the noise may be an order of magnitude lower. In this chapter, I confine myself to dc SQUIDs, fabricated from low transition temperature (T_c) superconductors. A detailed description of high-T_c SQUIDs can be found in the article by Koelle *et al*. [6]. A comprehensive account of SQUIDs and their applications can be found in the SQUID Handbooks [7,8]

In this chapter I describe the principles and operation of the dc SQUID, with an emphasis on its application to amplifiers in both the classical and quantum regimes. I begin, in Sec. 2, with a brief review of the resistively-shunted Josephson junction, with particular emphasis on the effects of noise and the observation of quantum fluctuations. In Sec. 3 I discuss the equations of motion for the dc SQUID, the current-voltage characteristics and noise in the classical and quantum regimes. Sections 4 and 5 are concerned with the theory and practice of SQUID amplifiers in the classical and quantum regimes, respectively. Chapter 6 contains a selection of applications of SQUID amplifiers.

2. The resistively shunted Josephson junction

2.1 Equation of motion: the classical Langevin equation

A Josephson junction [2] consists of two superconductors separated by a thin insulating barrier. Cooper pairs of electrons tunnel through the barrier, maintaining phase coherence in the process. The applied current, I, controls the difference $\delta = \phi_1 - \phi_2$ between the phases of the two superconductors according to the current-phase relation

$$I = I_0 \sin \delta, \tag{2.1}$$

where I_0 is the critical current, that is, the maximum supercurrent the junction can sustain. When the current is increased from zero, initially there is no voltage across the junction; for $I > I_0$ a voltage V appears, and δ evolves with time according to the voltage-frequency relation

$$\dot{\delta} = 2eV/\hbar = 2\pi V/\Phi_0. \tag{2.2}$$

A Josephson tunnel junction has a hysteretic current-voltage (I - V) characteristic. As the current is increased from zero, the voltage switches abruptly to a nonzero value when I exceeds I_0, returning to zero only when I is reduced to a value much less than I_0. This hysteresis must be eliminated for SQUIDs operated in the conventional manner, and one does so by shunting the junction with an external shunt resistance. The "resistively shunted junction" (RSJ) model [9,10] is shown in Fig.1(a). The junction has a critical current I_0 and is in parallel with its self-capacitance C and its shunt resistance R, which has a current noise source $I_N(t)$ associated with it. The equation of motion is

$$C\dot{V} + I_0 \sin \delta + V/R = I + I_N(t). \tag{2.3}$$

Neglecting the noise term for the moment and setting $V = \hbar\dot{\delta}/2e$, we obtain

$$\frac{\hbar C}{2e}\ddot{\delta} + \frac{\hbar}{2eR}\dot{\delta} = I - I_0\sin\delta = -\frac{2e}{\hbar}\frac{\partial U}{\partial \delta} \tag{2.4}$$

where

$$U = -(\Phi_0/2\pi)(I\delta + I_0\cos\delta). \tag{2.5}$$

One obtains considerable insight into the dynamics of the junction by realizing that Eq. (2.4) also describes the motion of a ball moving on the "tilted washboard" potential U. The term involving C represents the mass of the particle, the l/R term represents the damping of the motion, and the average "tilt" of the washboard is proportional to $-$I. For values of $I < I_0$, the particle is confined to one of the potential wells [Fig. 1(b)], where it oscillates back and forth at the plasma frequency [2] $\omega_p = (2\pi I_0/\Phi_0 C)^{1/2} [1- (I/I_0)^2]^{1/4}$. In this state $< \dot{\delta} > = 0$ and hence the average voltage across the junction is zero (< > represents a time average). As the current is increased to I_0, the tilt increases, and when I exceeds I_0, the particle rolls down the washboard; in this state $< \dot{\delta} > = 0$ is nonzero, and a voltage appears across the junction [Fig.1(c)]. As the current is increased further, $< \dot{\delta} >$ increases, as does V. For the nonhysteretic case, as soon as I is reduced below I_0 the particle becomes trapped in one of the wells, and V returns to zero. In this, the overdamped case, we require [9,10]

$$\beta_C \equiv (2\pi I_0 R/\Phi_0)RC = \omega_J RC \lesssim 1; \tag{2.6}$$

$\omega_J/2\pi$ is the Josephson frequency corresponding to the voltage I_0R.

We introduce the effects of noise by restoring the noise term in Eq. (2.4) to obtain the classical Langevin equation

$$\frac{\hbar C}{2e}\ddot{\delta} + \frac{\hbar}{2eR}\dot{\delta} + I_0 \sin\delta = I + I_N(t). \tag{2.7}$$

In the thermal noise limit, the spectral density of $I_N(t)$ is given by the Nyquist formula

$$S_I(f) = 4k_BT/R, \tag{2.8}$$

where f is the frequency. It is evident that $I_N(t)$ causes the tilt in the washboard to fluctuate with time. This fluctuation has two effects on the junction. First, when I is less than I_0, from time to time fluctuations cause the total current $I + I_N(t)$ to exceed I_0, enabling the particle to roll out of one potential minimum into the next. For the underdamped junction, this process produces a series of voltage pulses randomly spaced in time. Thus, the time average of the voltage is nonzero even though $I < I_0$, and the I - V characteristic is "noise-rounded" at low voltages [11]. Because this thermal activation process reduces the observed value of the critical current, there is a minimum value of I_0 for which the two sides of the junction remain coupled together. This condition is

$$I_0\Phi_0/2\pi \gtrsim k_BT, \tag{2.9}$$

where $I_0\Phi_0/2\pi$ is the coupling energy of the junction [2]. For T = 4.2 K, we find $I_0 \lesssim 0.2\ \mu A$.

The second consequence of thermal fluctuations is voltage noise. In the limit $\beta_C << 1$ and for $I > I_0$, the spectral density of this noise at a measurement frequency f_m that we assume to be much less than the Josephson frequency f_J is given by [12,13]

$$S_V(f_m) = \left[1+\frac{1}{2}\left(\frac{I_0}{I}\right)^2\right]\frac{4k_BTR_d^2}{R} \quad . \quad \left\{\begin{matrix}\beta_C << 1 \\ I > I_0 \\ f_m << f_J\end{matrix}\right\} \tag{2.10}$$

The first term on the right-hand side of Eq. (2.10) represents the Nyquist noise current generated at the measurement frequency f_m flowing through the dynamic resistance $R_d = dV/dI$ to produce a voltage noise (Fig. 2). The second term, $(1/2)(I_0/I)^2\ (4k_BT/R)R_d^2$, represents Nyquist noise generated at frequencies $f_J \pm f_m$ mixed down to the measurement frequency by the Josephson oscillations and the inherent nonlinearity of the junction. The factor mixing coefficient $(1/2)(I_0/I)^2$ vanishes for sufficiently large bias currents. The mixing coefficients for the Nyquist noise generated near harmonics of the Josephson frequencies $2f_J$, $3f_J$, ... are negligible in the limit $f_m/f_J << 1$.

2.1 The quantum Langevin equation

At sufficiently high bias current, the Josephson frequency f_J exceeds k_BT/h and Eq.(2.7) becomes a quantum Langevin equation for which the spectral density of $I_N(t)$ is $(2hf/R)\coth(hf/2k_BT)$. The spectral density of the voltage noise across the junction is

$$S_V(f_m) = \left[\frac{4k_BT}{R} + \frac{2eV}{R}\left(\frac{I_0}{I}\right)^2 \coth\left(\frac{eV}{k_BT}\right)\right] R_D^2. \qquad (2.11)$$

We have assumed that $hf_m/k_BT << 1$, so that the first term on the right-hand side of Eq. (2.11) remains in the thermal limit. Thus, quantum corrections [9] to the observed voltage noise become important in the limit $eI_0R/k_BT >> 1$ provided the term $(1/2)(I_0/I)^2$ is not too small. In the limit $hf >> 2k_BT$, the spectral density of the current noise in the resistor R reduces to the quantum value 2hf/R. In this limit, the second term on the right of Eq.(2.11), $(2eV/R)(I_0/I)^2 R_D^2$, represents noise mixed down from zero point fluctuations near the Josephson frequency.

2.3 Observation of quantum fluctuations

Zero point fluctuations were first observed in a current biased, resistively shunted Josephson junction using the circuit shown in Fig. 3. The voltage noise was measured at three frequencies by means of two LC-resonant circuits connected separately or in parallel to a low noise, room temperature amplifier. Measurements at the three frequencies allowed the subtraction of a small contribution of 1/f noise from the junction; the measured voltage noise and current noise of the preamplifier were also subtracted. In the low frequency limit, the spectral density of the voltage noise across a given tank circuit with inductance L_t was $Q^2S_v(0)$, where the quality factor $Q = \omega L_t/R_D$. Thus, the quantity $S_v(0)/R_D^2$ was independent of Q, and could be compared directly with the prediction

$$S_V(f_m)/R_D^2 = \left[\frac{4k_BT}{R} + \frac{2eV}{R}\left(\frac{I_0}{I}\right)^2 \coth\left(\frac{eV}{k_BT}\right)\right] \qquad (2.12)$$

using measured values of I_0, R, I, V, T and L_t. We note that the term $(2eV/R)(I_0/I)^2\coth(eV/k_BT)$ can also be written as $(4eV/R)(I_0/I)^2\{[\exp(2eV/k_BT) - 1]^{-1} + 1/2\}$, that is, essentially as the Planck energy + the zero point energy.

The results are shown in Fig.4. Figure 4(a) shows $S_v(0)/R_D^2$ vs. voltage (proportional to frequency) for the junction at 4.2 K. The open circles show total the measured noise and the solid circles show the noise after corrections. The solid line in the upper plots is the prediction of Eq. (2.12), whereas the dashed line is the prediction with the zero point term subtracted, that is, $(4eV/R)(I_0/I)^2\{[\exp(2eV/k_BT) - 1]^{-1}\}$. The lower set of plots show the mixed down noise, obtained by subtracting $4k_BT/R$ from the solid circles, and the solid line is the predicted value $(2eV/R)(I_0/I)^2\coth(eV/k_BT)$. The lower dashed line is the prediction with the zero point term removed. The plots in Fig. 4(a) show very clearly that the zero point term is required to fit the experimental data.

We can extract from the data the measured spectral density of the current noise $S_I(f)$ generated by the resistance R. We multiply each value of the mixed down noise by $2(I/I_0)^2$, and set $2eV = h\nu$. The results are plotted in Fig. 4(b) for 4.2 K (solid circles) and 1.6 K (open circles). The solid lines are the corresponding predictions of Eq. (2.12) with measured values of $\nu = 2eV/h$, R and T. The agreement between the predictions and the data is rather good, especially bearing in mind that there are no fitting parameters. The dashed lines represent the prediction in the absence of the zero-point term, and fall off dramatically at the higher frequencies.

These results demonstrate, first, the existence of a zero-point term in the spectral density of the current noise of a resistor in thermal equilibrium and, second, that these fluctuations give rise to the limiting voltage noise in a current-biased resistively shunted Josephson junction in the quantum limit for $I > I_0$. It should be emphasized that the observation of the zero point term is entirely due to the nonlinearity of the Josephson junction that mixes down high frequency noise near the Josephson frequency. One can think of the zero point fluctuations as randomly modulating the tilt of the washboard, a process that requires no energy but that modulates the rate at which the phase difference evolves with time. Furthermore, the good agreement between the data and the model predictions justifies the use of a quantum Langevin equation to calculate quantum noise in an overdamped, current-biased Josephson junction in the free running regime $I > I_0$. This gives us some confidence in the use of a quantum Langevin approach to calculate the noise in a dc SQUID in the quantum limit.

3. The dc SQUID

3.1 Equations of motion: the classical Langevin equation

Figure 5 shows the model for the dc SQUID. Two Josephson junctions are connected in parallel on a superconducting loop of inductance L. Each junction is resistively shunted to eliminate hysteresis on the I -V characteristics. When we current bias the SQUID into the voltage state and apply a monotonically increasing magnetic flux Φ, the critical current and I – V characteristic are modulated with period Φ_0. The SQUID is generally operated near the steepest region of the V - Φ curve, which occurs at about $(n + ½)\Phi_0/2$ where the flux-to-voltage transfer coefficient, $V_\Phi \equiv (\partial V/\partial \Phi)_I$ is a maximum. Thus, the SQUID produces an output voltage $\delta V = V_\Phi \delta\Phi$ in response to a small applied flux $\delta\Phi$, and is effectively a flux-to-voltage transducer.

Our goals are to calculate V_Φ, the spectral densities of the voltage noise $S_V(\Phi)$ and circulating current noise $S_J(f)$ and their cross correlation spectrum $S_{VJ}(f)$. The SQUID inductance is L, each junction has a critical current I_0, a self capacitance C and is shunted with a resistor R. The phase differences across the two junctions are δ_1 and δ_2, respectively, and the associated resistors have independent Nyquist noise currents I_{N1} and I_{N2}. The equations of motion are [16,17]:

$$V = (\hbar/4e)(\dot{\delta}_1 + \dot{\delta}_2), \tag{3.1}$$

$$J = (\Phi_0/2\pi L)[\delta_1 - \delta_2 - (2\pi\Phi/\Phi_0)], \tag{3.2}$$

$$(\hbar C/2e)\ddot{\delta}_1 + (\hbar/2eR)\dot{\delta}_1 = I/2 - J - I_0\sin\delta_1 + I_{N1}, \tag{3.3}$$

and

$$(\hbar C/2e)\ddot{\delta}_2 + (\hbar/2eR)\dot{\delta}_2 = I/2 + J - I_0\sin\delta_2 + I_{N2}. \tag{3.4}$$

Equation (3.1) relates the voltage to the average rate of change of phase; Eq. (3.2) relates the current in the loop, J, to $\delta_1 - \delta_2$ and to Φ; and Eqs. (3.5) and (3.6) are Langevin equations coupled via J. There are no analytical solutions for these coupled, nonlinear equations—at least in parameter ranges of practical interest. Rather, these equations have been solved numerically for a limited range of values of the noise parameter $\Gamma = 2\pi k_B T/I_0\Phi_0$, reduced inductance $\beta_L \equiv 2LI_0/\Phi_0$ and hysteresis parameter $\beta_c = 0$. For typical SQUIDs in the ^{4}He temperature range $\Gamma = 0.05$. Full details of Claudia Tesche's simulations can be found in ref. [16]. The first task is to compute the dependence of the critical current on applied flux, a purely static problem. A much more complicated calculation is to find the time averaged values of the I-V characteristic as a function of applied flux, from which one can compute the time-averaged voltage V vs. Φ, and hence find V_Φ. One also computes the current J circulating around the SQUID loop.

3.2 Current-voltage characteristic, flux-to-voltage transfer function and noise

As an example of the results, Fig. 6 shows the time averaged I-V characteristic of a SQUID for three values of magnetic flux. Noise rounding at low voltages is clearly visible. Figure 7 shows three key results: V_Φ, $S_V(0)$ and flux noise $S_\Phi^{1/2}(0)$ vs. I/I_0. Figure 7(a) shows that V_Φ peaks as a function of bias current, at a value that depends on the applied flux. For each value of flux, the peak occurs at the maximum value of the dynamic resistance; the peak is highest when the flux is $\Phi_0/4$. The spectral density of the noise voltage was computed as a function of bias current at fixed flux, and found to be white at frequencies well below the Josephson frequency. Figure 7(b) shows that for each value of flux the noise spectral density peaks smoothly at the value of I where V_Φ is a maximum. These results are combined in Fig. 7(c). in which we plot the ratio $S_V^{1/2}(0)/V_\Phi$ to yield the flux noise $S_\Phi^{1/2}(0)$ vs. bias current for three values of flus. We note that the minimum in flux noise is substantially broader in bias current that the peaks in voltage noise and transfer function, and that the lowest flux noise occurs at $\Phi_0/4$.

A convenient way of comparing the flux noise in SQUIDs with different parameters is in terms of the noise energy per unit bandwidth

$$\varepsilon = S_\Phi(0)/2L. \qquad (3.5)$$

From a series of simulations, one finds that the noise energy has a minimum when $\beta_L = 1$. For $\Gamma = 0.05$, $\beta_C = 1$, $\Phi = (2n = 1)\Phi_0/4$ and at the value of I at which V_Φ is a maximum, the optimized results can be summarized as follows:

$$V_\Phi \approx R/L, \qquad (3.6)$$

$$S_V(0) \approx 16k_BTR, \qquad (3.7)$$

$$S_\Phi(0) \approx 16k_BTL^2/R, \qquad (3.8)$$

$$\varepsilon \approx 9k_BTL/R \approx 16k_BT(LC)^{1/2}. \qquad (3.9)$$

Equation (3.6) shows that V_Φ can be written as a characteristic frequency, and Eq. (3.7) shows that the voltage noise spectral density is about 8 times the Nyquist noise in a resistance R/2 (the parallel resistance of the two shunt resistors). To obtain the last expression in Eq. (3.9), we set $R = (\Phi_0/2\pi I_0C)^{1/2}$ ($\beta_c = 1$). The resulting expression shows that the noise energy scales with T, reflecting its origin in Nyquist noise, and inversely with the characteristic frequency $\sim (LC)^{-1/2}$. Thus, in the classical limit, reducing the temperature reduces the noise energy. Furthermore, “smaller is better”—lowering the loop inductance and junction capacitance will reduce the noise energy. These results have been found to be good predictors for the performance of practical SQUIDs.

As we shall see in our discussion of amplifiers, however, the noise energy is not a complete specification of the SQUID because it does not account fully for the circulating current noise. Claudia calculated the current noise, and its spectral density $S_J(0)$ is plotted in Fig. 8(a). For fixed flux, the current noise peaks as a function of bias current. As the flux is increased from zero, the peak current noise increases, diverging at $\Phi_0/2$. For a SQUID with $\beta_L = 1$, $\Gamma = 0.05$ and $\Phi = (2n+1)\Phi_0/4$, the peak spectral density of the current noise is [17]

$$S_J(f) \approx 11\ k_BT/R. \qquad (3.10)$$

Furthermore, the current noise is partially correlated with the voltage noise across the SQUID, as shown in Fig. 8(b). For the same parameter values, the peak cross-spectral density is [17]

$$S_{VJ}(f) \approx 12\ k_BT. \tag{3.11}$$

The correlation arises because the current noise generates a flux noise which, in turn, contributes to the total voltage noise across the junction, provided $V_\Phi \neq 0$.

3.3 Practical dc SQUIDs

Modern dc SQUIDs are made from thin films with the aid of either photolithography or electron-beam lithography, and come in a great variety of designs. A widely used design that is available commercially was introduced by Mark Ketchen and Jeffrey Jaycox [18], and is shown in Fig. 10. The SQUID body consists of a square washer with a spiral, superconducting input coil deposited on it with an intervening insulating layer. Such devices are typically fabricated in batches of several hundred on oxidized silicon wafers. Except for the resistive shunts, the entire structure is made of Nb. The junctions are patterned from a Nb/AlOx/Nb [19]. In this process, following the deposition of the Nb base electrode and a thin Al layer, the Al is oxidized in a reduced pressure of oxygen and the Nb counter electrode is deposited. The entire trilayer is formed without removing the wafers from the controlled atmosphere of the sputter system. The junction areas are defined by anodizing a small ring of the counter electrode, and the base electrode is etched to form the SQUID washer. In subsequent operations, one adds the Nb layer that forms the input and flux modulation coils and makes the connection to the counter electrode, the shunt resistors (typically Mo or Pd), and the final Nb layer that connects the innermost turn of the input coil. The insulation between each layer is usually SiO_2, and patterning is performed with reactive ion etching. Typical loop inductances are 100 to 400 pH and the shunt resistances are a few ohm.

Design guidelines for the square washer SQUID were given by Ketchen and Jaycox[18], who showed that a square washer (with no slit) with inner and outer edges d and w has an inductance L (loop) = 1.25 $\mu_0 d$ in the limit w >> d. They gave the following expressions for the inductances of the SQUID and spiral coil, L and L_i, and for the mutual inductance, M_i, between them:

$$L = L\ (\text{loop}) + Lj, \tag{3.12}$$

$$L_i = n^2(L-L_j) + L_s, \tag{3.13}$$

$$M_i = n(L-L_j) \tag{3.14}$$

$$\alpha^2 = (1-L_j/L)/[1+L_s/n^2(L-Lj)]. \tag{3.15}$$

Here, Lj is the parasitic inductance associated with the junctions and slit, n is the number of turns on the input coil and L_s is the stripline inductance of this coil, which is generally much smaller than L_i for n $\gtrsim$ 20. Measured parameters are generally in good agreement with these predictions.

4. Low frequency SQUID amplifiers

4.1 Noise Temperature

Before delving into the theory of SQUID amplifiers, it is convenient first to introduce the concept of noise temperature T_N. For simplicity, we consider a field effect transistor (FET) amplifier which at low frequencies has a high input impedance and does not load the source. Its voltage gain is –A. Referred to its input terminals, the FET has a *virtual* voltage noise e_n and an uncorrelated *actual* current noise i_N that develops a voltage noise $i_N R_i$ across a resistance R_i connected across its input. Our goal is to determine the optimum value of R_i that minimizes T_N.

The voltage noise at the amplifier output is $V_o = -A(e_N + i_N R_i)$. Since e_N and i_N are uncorrelated, the spectral density of this noise is $S_V^o = A^2(S_e + S_i R^2)$. We introduce T_N as the temperature at which the Nyquist noise associated with R_i would produce the equivalent output noise spectral density, $4k_B T R_i A^2$. Equating these two quantities gives $T_N = (S_e/R_i + S_i R_i)/4k_B$. Finally, differentating with respect to R_i yields the optimized values

$$R_i^{opt} = (S_e/S_i)^{1/2} \tag{4.1}$$

and

$$T_N^{opt} = (S_e S_i)^{1/2}/2k_B. \tag{4.2}$$

4.2 SQUID amplifier theory, noise and optimization

We now discuss the theory, operation and performance of SQUID amplifiers at frequencies up to, say, 100 MHz. In this frequency range, we can treat the amplifier in the lumped circuit approximation or “op-amp” approximation—in fact, as we shall see, the SQUID amplifier is in many ways the dual of a semiconductor operational-amplifier. We recognize at the beginning that the SQUID is a complicated, highly nonlinear device. In particular, the capacitance between the input coil and the SQUID washer may attenuate the coupling of the SQUID to the input circuit at the Josephson frequency. In the limit in which there is no attenuation or very high attenuation, the calculations of the influence of the SQUID on the input circuit and vice versa are straightforward. In practice, the coupling is likely to be somewhere between the two limits, so that the mutual influences are nontrivial calculations. Consequently we consider a rather simplified model based on more detailed publications [20 – 22].

If we imagine “looking” into the input terminals of a coil to SQUID we will "see" a dynamic impedance *Z* in the SQUID loop that can be written in the form [19]

$$1/\mathcal{Z} = 1/j\omega\mathcal{L} + 1/\mathcal{R}, \tag{4.3}$$

where $j = \sqrt{-1}$. The dynamic inductance $\mathcal{L}$ and dynamic resistance $\mathcal{R}$ are not simply related to L and R, but vary with bias current and flux; for example, $1/\mathcal{L}$ is zero for certain values of flux. The terms $\mathcal{L}$ and $\mathcal{R}$ introduce additional inductances and into the input circuit; furthermore, the input circuit renormalizes L and V_Φ. Strictly speaking, we should take these corrections into account. It turns out, however, that for a tuned amplifier with a reasonably high quality factor, we can neglect these terms, as we shall see.

The configuration of a tuned amplifier is shown in Fig. 10(a). An input voltage source $V_i(t)$ with source resistance R_i is connected in series with a capacitor C_i, the input coil of a SQUID and some stray inductance L_s. The output voltage from the SQUID is V_o. In general, the presence of the input circuit modifies all the SQUID parameters, including the noise terms [20, 21]. By the same token, the dynamic impedance of the SQUID is reflected into the input circuit. If the *effective* coupling coefficient between the source and the SQUID is sufficiently weak, however, we can neglect these mutual interactions. For the purpose of illustration, we assume a SQUID with given values of M_i, L_i, L, V_Φ, $S_V(f)$, $S_J(f)$ and $S_{VJ}(f)$, and find the values of C_i and R_i that optimize the noise temperature.

4.2.1 Tuned amplifier: on resonance

In the weak coupling limit, the SQUID noise voltage is represented as a *virtual* current i_N in the input circuit V_N/M_iV_Φ. The noise current J_N in the SQUID loop induces an *actual* voltage $e_N = -j\omega M_iJ_N$ into the input circuit [Fig. 10(b)]; since on resonance the impedance of the input circuit is R_i, the current generated is $-j\omega M_iJ_N/Ri$. It is important to note that this current is in quadrature with J_N and thus with the contribution that J_N makes to the voltage noise across the SQUID. We note that the noise terms e_N and i_N are the dual of those for an FET, for which the current is actual and the voltage is virtual. Inserting the spectral densities of e_N and i_N into Eq.(4.1) immediately yields

$$R_i^{res} = \omega M_i^2V_\Phi(S_J/S_V)^{1/2} \approx \alpha^2\omega L_i, \tag{4.4}$$

and

$$T_N^{res} = (S_VS_J)^{1/2}/k_BV_\Phi \approx 18fT/ V_\Phi \approx 2f\varepsilon(f)/k_B, \tag{4.5}$$

where we have used Eqs. (3.6), (3.7) and (3.10). We note that the cross spectral term S_{VJ} drops out of T_N^{res}. This is because the current induced into the input circuit is in quadrature with J_N, and is thus uncorrelated with the contribution of J_N to the voltage noise across the SQUID. Furthermore, as noted earlier, although $\varepsilon(f)$ does not fully characterize a SQUID amplifier, within the framework of the model, it does enable one to predict T_N.

We now introduce the quality factor of the tuned circuit

$$Q = \omega(L_i + L_s)/R_i \approx (L_i + L_s)/\alpha^2L_i{}^2 = 1/\alpha_e{}^2, \tag{4.6}$$

where we have used Eq.(4.4). Here,

$$\alpha_e{}^2 = \alpha^2L_i/(L_i + L_s) = M_i^2/L(L_i + L_s) \tag{4.7}$$

is the *effective* coupling coefficient between the SQUID and the input circuit. Since $Q/\alpha_e{}^2 \approx 1$, we see that even moderately high-Q imply that $\alpha_e{}^2$ is small, thereby justifying the assumption that we can neglect the mutual interaction of the SQUID and input circuit.. One also finds

One can readily calculate the gain on resonance. For $\alpha^2 << 1$, an input signal V_i produces an output voltage $V_o \approx (V_i/R_i^{res})M_iV_\Phi$. Thus, the square of the voltage gain is given by

$$G_v \approx M_i^2V_\Phi{}^2/(R_i^{res})^2 \approx (R/R_i^{res})(V_\Phi/\omega), \tag{4.8}$$

where we have used Eqs.(3.6) and (4.4). We see that G_v is the ratio of the characteristic frequency V_Φ to the signal frequency f, and in this sense is reminiscent of a parametric amplifier. The dc SQUID mixes up the signal to a high frequency, and down converts it to the signal frequency with gain.

4.2.2 Tuned amplifier: optimized noise temperature

Operating a SQUID tuned amplifier at the resonant frequency, however, does not give the lowest noise temperature, and we now consider the off-resonance case. In the weak coupling limit, the noise current J_N induces a voltage $-j\omega M_iJ_N$ into the input circuit, and hence a current $-j\omega M_iJ_N/Z_i$, where

$$Z_i \approx R_i + j\omega(L_i + L_s) + 1/j\omega C_i \tag{4.9}$$

is the impedance of the input circuit. In general, this current is not in quadrature with J_N, since the input circuit has a complex impedance. This noise current, in turn, induces a flux in the SQUID loop and finally a voltage $-jM_i^2J_NV_\Phi/Z_i$ across the SQUID. Thus, the noise voltage across the SQUID in the presence of the input circuit is

$$V_N' = V_N - j\omega M_i^2 J_N V_\Phi/Z_i, \tag{4.10}$$

where V_N is the noise voltage of the bare SQUID, which we assume to be unchanged by the input circuit in the limit of small α. The spectral density of V_N' is found to be

$$S'_V(f) = S_V(f) + \omega^2 M_i^4 V_\Phi^2 S_J(f)/|Z_i|^2 - \{2\omega M_i^2 V_\Phi[\omega(L_i + L_s) - 1/\omega C_i]S_{VJ}(f)\}/|Z_i|^2. \tag{4.11}$$

We now apply a sinusoidal input signal frequency $\omega/2\pi$, with a mean-square amplitude $<V_i^2>$. The mean-square signal at the output of the SQUID is

$$<V_o^2> = M_i^2 V_\Phi^2 <V_i^2>/|Z_i|^2. \tag{4.12}$$

The signal-to-noise ratio is

$$S/N = <V_o^2>/ S'_V(f)\, B \tag{4.13}$$

in a bandwidth B. We introduce the noise temperature T_N for the amplifier by setting S/N = 1 with $<V_i^2> = 4k_BT_NR_iB$. This procedure implies that the output noise power generated by the SQUID is equal to the output noise power generated by the resistor R_i when it is at a temperature T_N. We optimize T_N with respect to R_i and C_i for a given value of L_i to find

$$R_i^{opt} = [\alpha^2\omega(L_i + L_s)LV_\Phi/S_V](S_VS_J - S_{VJ}^2)^{1/2}, \tag{4.14}$$

$$1/\omega C_i^{opt} = \omega(L_i + L_s)(1 + \alpha^2 S_{VJ} L V_\Phi)/S_V, \tag{4.15}$$

and

$$T_N^{opt} = (\pi f/k_B V_\Phi)(S_V S_J - S_{VJ}^2)^{1/2}. \tag{4.16}$$

We note from Eq.(4.16) that the optimum noise temperature occurs off-resonance. For the values of the spectral densities given in Eqs. (3.7), (3.10) and (3.11), $T_N^{opt}/T_N^{res} \approx 0.4$.

As a final remark, we note that this theory is concerned only with the noise temperature of the amplifier itself. Nyquist noise from the input resistor may add a contribution that exceeds the amplifier noise. When the value of T_N is well below T, the optimization procedure outlined above does not necessarily give the lowest system noise.

4.3 Experimental configuration, operation and performance

Hilbert and Clarke [22] made several radiofrequency amplifiers with both tuned and untuned inputs, flux biasing the SQUID near $\Phi = (2n + 1)\Phi_0/4$. There was no flux-locked loop. The measured parameters were in good agreement with predictions. For example, for an amplifier with $R \approx 8\ \Omega$, $L \approx 0.4$ nH, $L_i \approx 5.6$ nH, $M_i \approx 1$ nH and $V_\Phi \approx 3\text{x}10^{10}$ sec^{-1} at 4.2 K, they found G = 18.6 ± 0.5 dB and T_N = 1.7 ± 0.5 K at 93 MHz. The predicted values were 17 dB and 1.1 K, respectively.

5. High frequency SQUID amplifiers: the quantum limit

5.1 Noise and optimization in the quantum limit

For a linear, phase preserving amplifier, the quantum limited noise temperature is given by

$$T_Q = hf/2k_B. \qquad (5.1)$$

More generally, convenient way of expressing the noise temperature is in terms of Caves' added noise number A [23]:

$$T_N = Ahf/k_B. \qquad (5.2)$$

Clearly, A = ½ for a quantum limited amplifier.

The noise temperature for an optimized SQUID tuned amplifier at T = 0 was computed by Roger Koch [24]. The approach was to replace the thermal noise currents in Eqs.(3.3) and (3.4) with quantum noise currents with spectral density 2hf/R, compute the quantities V_Φ, S_V, S_J, and S_{VJ} in the quantum regime and optimize the parameters for lowest noise temperature. The results are summarized in Fig. 11, which shows seven computed and derived quantities vs. flux at a constant bias current for R = 40 Ω, β_L = 1 and three values of β_c. The peak values of V_Φ, S_V, S_J and S_{VJ} for $\beta_c = 1$ are substantially higher than those for $\beta_c = 0.25$ and 0.5, reflecting a higher dynamic resistance. The values of values of V_Φ and S_V were used to compute ε [Fig. 11(f)], which for the lower values of β_c has a minimum below ħ. Finally, V_Φ and the three noise terms were used in Eqs.(4.16) and (5.2) to compute A, which had a minimum value of 0.5, off resonance. The estimated computational accuracy was ± 15 %. Needless to say, the prediction that the SQUID with a tuned input circuit should ideally be a quantum limited amplifier assumes the validity of both the quantum Langevin equation and the optimization proceedure developed for the thermal limit.

To demonstrate quantum limited amplification, one requires $T_Q > T$. For a lowest practical operating temperature of 20 mK, this implies that the signal frequency should be greater than about 0.5 GHz. At the time the theory was developed (1981), SQUIDs were invariably used at much lower frequencies, and there was no motivation to develop gigahertz devices.

5.2 The microstrip SQUID amplifier

5.2.1 Principles, gain and tuning

The original motivation to develop a SQUID amplifier with high gain and low noise at gigahertz frequencies was the need for such a device for the axion detector (Sec. 6.1). The immediate challenge with the square washer SQUID—which has been so successful at lower frequencies—is the parasitic capacitance between the coil and the washer that rolls off the response above a few tens or at most 100 MHz. This problem was overcome by Michael Mück who moved one wire so that the input signal—instead of being connected to the two ends of the input coil—was connected to one end of the input coil and the washer [25]. Thus, the signal is propagated along the microstrip formed by the coil inductance and its capacitance to the washer. When the length of the coil corresponds to $\lambda/2$, where λ is the signal wavelength on the microstrip, one expects to see a resonance that couples the signal strongly to the SQUID.

A microstrip consists of a superconducting strip of width w separated from an infinite superconducting sheet by an insulator with dielectric constant ε and thickness d. We assume that the thicknesses of the two superconductors are much greater than the superconducting penetration depth λ_s, and that $w >> d$. The capacitance and inductance per unit length of the microstrip are given by $C_s = \varepsilon\varepsilon_0 w/d$ ($\mathrm{Fm^{-1}}$) and $L_s = (\mu_0 d/w)(1 + 2\lambda_s/d)$ ($\mathrm{Hm^{-1}}$) [26]. Here, $\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{Fm^{-1}}$ and $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{Hm^{-1}}$ are the permittivity and permeability of free space, and $c = 1/(\varepsilon_0\mu_0)^{1/2} = 3 \times 10^8\ \mathrm{ms^{-1}}$ is the velocity of light in vacuum. The factor $(1 + 2\lambda_s/d)$ accounts for the penetration of the magnetic field into the (identical) superconductors. The velocity of an electromagnetic wave on the microstrip is thus given by $\bar{c} = c/[\varepsilon(1 + 2\lambda_s/d)]^{1/2}$, and its characteristic impedance by $Z_s = (L_s/C_s)^{1/2} = (d/w)[\mu_0(1 + 2\lambda_s/d)/\varepsilon\varepsilon_0]^{1/2}$. For a microstrip of length ℓ with its two ends either open or terminated with resistances greater than Z_0, the fundamental frequency occurs when $\ell = \lambda/2$ [26],

$$f_0(L_s) = c/2\ell[\varepsilon(1 + 2\lambda_s/d)]^{1/2}. \tag{5.3}$$

The resonant frequency of the microstrip SQUID, however, is complicated by the inductive loading produced by the SQUID. Equation (3.13) implies that L_s should be replaced by n^2L/ℓ, where we have assumed that $L >> L_s$ and neglected L_j, resulting in the resonant frequency [27]

$$f_0(n^2L) = c/2n(\ell LC_s)^{1/2}. \tag{5.4}$$

Figure 12(a) shows the circuit of the microstrip SQUID amplifier (MSA). In the first experiments, a sweep oscillator was coupled to the MSA input via a room-temperature, 100-dB attenuator and a cold 20-dB attenuator that prevented noise from the generator from saturating the SQUID. The cold attenuator also presented an impedance of 50 Ω to both the input coaxial line and the MSA. A second cold, 4-dB attenuator coupled the output of the SQUID to a room-temperature postamplifier. The gain of the system excluding the MSA was calibrated by disconnecting the MSA and connecting together the input and output attenuators. All gain measurements were referred to the baseline so obtained. Figure 12(b) shows gain vs. frequency for four MSAs with progressively shorter coils. The peak gain was about 18 dB, and occurred at progressively higher frequencies as the coil length was reduced. These frequencies were given approximately by Eq. (5.4).

In many applications, it is desirable to tune the frequency at which the maximum gain occurs. Tuning is accomplished by connecting a GaAs varactor diode across the otherwise open end of the coil and the washer [28]. The capacitance of the diode can be varied by changing the value of the reverse bias voltage. Changing the capacitance modifies the phase shift of the electromagnetic wave when it is reflected, thereby increasing or decreasing the effective length of the microstrip and lowering or raising the peak frequency. In the experiments, the capacitance of two diodes in parallel (to increase the tuning range) could be varied from 1 to 10 pF by changing the bias voltage from 1 V to −22 V. The diodes were connected in series with a capacitor to avoid applying a static voltage to the microstrip. The gain for optimized current and flux biases for a SQUID with 31 turns is shown in Figure 13 for 9 values of diode capacitance. We see that the peak frequency is progressively lowered, from 195 MHz to 117 MHz, as the capacitance is increased. The maximum gain is constant at about 28 dB over this range. In the absence of the varactor diodes, the peak frequency is about 200 MHz. The dependence of the peak frequency on the varactor capacitance is in reasonable agreement with a simple model [28]. The presence of the varactors increases the gain, most likely by increasing the degree of positive feedback from the output to the input.

5.2.2 Scattering parameters and input matching

To maximize the gain of the MSA, it is essential to know its input impedance, which is generally complex. A two-port network can be described by a scattering matrix that relates the voltage V^+ incident at one port with the voltage V^- reflected from a second port [29]. The scattering parameter is defined as $S_{ij} = V_i^-/V_j^+$, where S_{11} is the input reflection coefficient with the output port terminated by a matched load, and S_{21} is the forward gain. Figure 14 shows the configuration for a reflection measurement of S_{11} [29]. The vector network analyzer (VNA) and the various cables were calibrated by replacing the input of the MSA, in turn, with an open-circuit, a short-circuit and a 50-Ω resistor.

Measurements were made on a MSA with the following parameters. For the SQUID, $L \approx 450$ pH, $I_0 \approx 2$ μA, $C \approx 0.2$ pF, $R \approx 20$ Ω, $\beta_L \approx 0.9$ and $\beta_c \approx 0.5$; for the coil, $n = 11$, $\ell \approx 15$ mm, $w \approx 5$ μm, $d \approx 400$ nm and $\varepsilon \approx 5.5$. Figure 15 shows S_{11}, converted to input impedance, versus measurement frequency with a flux bias $\Phi_0/4$. For a low-loss transmission line, these resonance curves can be described [29] by the input impedance

$$Z_{in} = Z_0[Z_L + Z_0\tanh(\gamma\ell)]/[Z_0 + Z_L \tanh(\gamma\ell)]. \qquad (5.5)$$

Here, Z_0 is the characteristic impedance, Z_L is the terminating impedance, and $\gamma \equiv \alpha + i\beta$ is the complex propagation constant. The data are an excellent fit to Eq.(5.5) with $Z_L = \infty$, α =1.9 Np/m, β =209 m^{-1}, and the resonant frequency f_r = 506.2 MHz. For the given microstrip geometry, one finds $Z_0 \approx 16$ Ω, which is reasonably consistent with the measured value of 14 Ω, and $\bar{c} \approx 0.33c$ [26], which predicts a resonant frequency of about 3 GHz. If, instead, we use Eq.(5.4), we find a resonant frequency of about 700 MHz, which is not too far removed from the measured value. Since the effect on f_r is much greater than on Z_0, we conclude that the loading is in the form of a lumped—rather than distributed—inductance. We note that changes in both the current and flux biases affect the S-parameters.

With $Z_L = \infty$, the impedance can be equivalently described by the parallel *ReLeCe* circuit shown in Fig. 14(b); C_1 represents the static capacitance of the line. This model gives a direct measure of the circuit parameters [27]. Figure 16 shows the variation of f_r, V, Z_0, C_e, L_e, and R_e with bias flux. The variation is very mixed. The sign of V_Φ is positive for $\Phi < \Phi_0/2$ and negative for $\Phi > \Phi_0/2$. Both Z_0 and L_e are roughly asymmetric about $\Phi_0/2$. On the other hand, f_r, C_e and R_e exhibit no evident systematic behavior.

We determined the forward scattering parameter S_{21}—essentially the amplitude gain—by connecting the output signal from the SQUID to the room temperature postamplifier and measuring it with the VNA [Fig. 14(a)]. We calibrated the gain by replacing the MSA with a short. The real and imaginary parts of S_{21} are shown in Fig. 17(a). From the circuit model in Fig. 17(b), we predict S_{21} to be $M_i V_\Phi i_L/V_i$, where M_i is the mutual inductance between the coil and the SQUID, i_L is the rf current in the microstrip, and V_i is the input voltage. This circuit model is as before with the addition of a voltage source with an impedance R_s of 50 Ω and a coupling capacitor C_c. The solid lines are predictions from the expression for S_{21} using the values C_1 = 3.2 pF, C_c = 222 pF, L_e= 2.4 nH, C_e = 40.9 pF, and R_e=400 Ω from the measurement of S_{11}, and fitting an overall scale factor corresponding to the transimpedance $M_i V_\Phi$ = 23.0 Ω. The static values V_Φ=40 μ$V/_\Phi 0$ and M_i = 3.5 nH yield a transimpedance of 68 Ω, about a factor of 3 higher. This implies a reduction in M_i or V_Φ from its low-frequency value; possibly both are reduced.

The central goal of these measurements is to determine the coupling of the input circuit to the MSA required to optimize the gain and noise temperature. The S_{11} results indicate that the intrinsic quality factor Q of the MSA at 4.2 K is typically 40–80. The Q values found from the S_{21} measurements with a source impedance of 50 Ω, however, are typically 5–20, implying that the source impedance significantly damps the resonator. Decreasing the coupling between the source and the MSA would increase Q, but at the same time, reduce the signal coupled to the resonator. The problem of coupling a resonator to a real source impedance is solved using critical coupling, that is, matching the real impedance at resonance to the source impedance by means of a series capacitor. The circuit model resulting from the S_{11} measurement can be used to estimate the required series coupling capacitance [29] $C_c = (16p^2 f_r^3 R_s^2 R_e C_e)^{-1/2}$. For the circuit parameters of a different MSA with nine turns, $f_r = 0.816$ GHz, $R_e = 714\ \Omega$ and $C_e = 4.4$ pF, we find $C_c \approx 1.4$ pF. The measured gain for three values of C_c are shown in Fig. 18. The maximum gain occurs for $C_c = 2.2$ pF. The resonator is clearly overcoupled for $C_c = 10$ pF and undercoupled for $C_c = 0.5$ pF. These results are in good agreement with the predictions from the equivalent circuit model.

These measurements demonstrate that the important properties of the MSA can be represented by a low-loss transmission line leading to an equivalent circuit model. By measuring the input impedance with the coil open circuited, one can predict the maximum gain and frequency response. One can also design the input circuit to give maximum gain by critically coupling the source to the microstrip resonator. Needless to say, the model parameters are strongly dependent on the values of I_o, β_c, β_L, and R; Since I_0 may change with temperature, the S parameters should ultimately be measured at the desired operating temperature.

5.2.3 Noise temperature

Numerous measurements have been made of the noise temperature of the MSA. For brevity, I shall describe only the most recent measurements, carried out at millikelvin temperatures [31], with the MSA design based on the results of S-parameter measurements. The inner and outer dimensions of the Nb washer were 0.2 and 1 mm, corresponding to a loop inductance (including the slit) of about 400 pH. Each of the two Nb-AlOx-Nb Josephson tunnel junctions, with dimensions of $2 \times 2\ \mu m^2$, had the following approximate parameters: $I_0 \approx 4\ \mu A$, $C \approx 0.2$ pF, $R \approx 16\ \Omega$, $\beta_L \approx 1.6$, $\beta_C \approx 0.6$ and $V_\Phi \approx 100\ \mu V/\Phi_0$. The 8-turn coil, with a 5-μm width and 15-μm pitch, produced a resonance varying from 620 to 645 MHz, depending on the SQUID static current and flux biases. At low temperatures, dissipation in the resistive shunts typically raises the electron temperature to 120 – 150 mK, thereby increasing the Nyquist noise [32]. To reduce this temperature, we connected a $500 \times 500\ \mu m^2$ cooling fin [32], fabricated from 500-nm-thick Au-Cu alloy, to a corner of each shunt resistor (Fig. 19).

Figure 20 shows a schematic of the experiment. The MSA, together with its bias and coupling circuitry, was housed in a copper box, lead-plated on the inside, that shielded against radiofrequency (rf) interference and changes in ambient magnetic fields. The box, surrounded with a Cryoperm shield, was mounted on the mixing chamber of a dilution refrigerator, and cooled to temperatures ranging from 40 mK to 1 K. All bias lines were heavily filtered. A GaAs HEMT (High Electron Mobility Transistor) post-amplifier with a gain of 18 dB and a noise temperature T_p of about 1.4 K was installed in the helium bath. To reduce out-of-band noise from the HEMT that could couple to the MSA we inserted a coaxial low-pass filter, with a cutoff frequency of 1 GHz, between the MSA output and HEMT input. The MSA output was matched to the 50-Ω impedance of the filter with a lossless L-match, consisting of a 12-pF off-chip capacitor [C_m in Fig. 20] connected across the series inductance (L_m, approximately 5 nH) of the on-chip output leads of the MSA. The gain of the MSA was optimized by means of a 2.2 pF capacitor (C_c) in series with the input resonator, critically coupling it to the 50-Ω source impedance. The gain of the MSA was determined by connecting a signal from a vector network analyzer (VNA) to the input of the MSA via a cold directional coupler. The output power was amplified and measured with the VNA. A calibration run determined the loss and electrical length of the signal path with the MSA replaced with a short.

The value of T_N was measured over a wide frequency range using the hot-cold load technique. With the MSA input connected to a 50-Ω resistor, the output power consists of the noise power contributed by the amplifier $P_N = Gk_BT_NB$. , and the amplified Nyquist noise power of the resistor $P = ½GhfBcoth(hf/2k_BT) \approx Gk_BTB$. The hot-cold load technique involves two measurements of the output power, $P_1 = Gk_B(T_1 + T_N)B$ and $P_2 = Gk_B(T_2 + T_N)B$, with the 50-Ω source at different temperatures T_1 and T_2. The ratio $R \equiv P_2/P_1$ yields $T_N = (T_2 - RT_1)/(R - 1)$. The variable temperature source consisted of a 50-Ω, SMA termination, a wire-wound heater and a RuOx thermometer embedded in a block of oxygen-free, high-conductivity copper. The temperature of the block was regulated with feedback. The optimal current and flux bias points were determined with automated scans.

Noise power spectra were acquired with the 50-Ω load resistor at 100 and 300 mK; typically 5000 spectra were averaged together to produce the values of P_1 and P_2. At each frequency, the ratio P_2/P_1 was used to determine the system noise temperature T_S. Obtaining T_N from T_S requires two corrections. The first is to subtract Tp/G to correct for the HEMT noise, taking into account the 0.3 dB loss between the MSA and the HEMT. This reduction is about 12 mK at resonance. The second correction is for the measured insertion loss of the directional coupler and the cable loss between the 50-Ω resistor and the MSA, about 0.2 dB, corresponding to a 2-mK correction to T_N.

Figure 21(a) shows the measured gain and lowest measured noise temperature versus frequency at 45 mK. The minimum noise temperature was 48 ± 5 mK, a factor of 1.6 above the value T_Q = 29.4 mK at 612 MHz, for a gain of 20.4 dB and a bandwidth of 21 MHz; the corresponding added noise number A = $(T_N/T_Q) - ½$ = 1.1. This value of T_N is a factor of about 30 lower than that of the best GaAs HEMT amplifiers. An interesting feature of Fig. 21(a) is the frequency dependence of T_N. The lowest value of T_N, 48 ± 5 mK, occurs slightly below resonance at about 612 MHz, whereas the value on resonance is 66 ± 5 mK. This behavior was seen consistently in three separate runs of the experiment, and is in qualitative agreement with the predictions of Eq.(4.16).

Finally, the entire process was repeated as the temperature was progressively raised to 1000 mK. Figure 21(b) shows the minimum measured T_N versus T. We see that T_N scales linearly with decreasing temperature until saturating at a value above T_Q at $T \approx 100$ mK. In separate experiments at 100 kHz, it was found that the flux noise also flattened out at $T \approx 100$ mK, demonstrating that hot-electrons limited the ultimate noise temperature [32].

5.3 Other dc SQUID amplifiers

There are at least two other approaches to using the dc SQUID as an amplifier, both of which separate the role of the resonator from the structure of the SQUID [33, 34]. [To be continued.]

6. Applications

6. Applications

6.1 The axion detector: The search for cold dark matter

The original impetus for the development of the MSA was the need for a lower-noise amplifier for the axion detector at Lawrence Livermore National Laboratory (LLNL), and I briefly describe this application [35].

There is overwhelming cosmological evidence that about 22% of the mass of the universe is cold dark matter (CDM); the corresponding density is approximately 0.45 $\mathrm{MeVc^{-2}mm^{-3}}$. Two leading contenders for CDM are the WIMP (Weakly Interacting Massive Particle), which supersymmetry theories predict to have a mass of 10–100 GeV, and the axion. The axion was originally postulated to satisfy a requirement in particle theory. The upper limit measured for the electric dipole moment on the neutron is 10^{11} times smaller than the value expected from the Standard Model of particle physics. This result implies that *CP* conservation is vastly stronger than predicted. (Here, *C* is charge conjugation and *P* is parity inversion.) Peccei and Quinn [36] extended the Standard Model to suppress strong *CP* violation, and subsequently Weinberg [37] and Wilczek [38] independently postulated the axion—a neutral, spinless particle—to resolve the *CP* problem. It is predicted that the rest mass m_a of the axion lies between 1 $\mu\mathrm{eVc^{-2}}$ and 1 $\mathrm{meVc^{-2}}$ (corresponding to frequencies of approximately 240 MHz and 240 GHz). For $m_a = 1\ \mu\mathrm{eVc^{-2}}$, the corresponding number density is $n_a = 4.5 \times 10^{11}\ \mathrm{mm^{-3}}$.

In 1983 Sikivie [39, 40] showed that in the presence of a high magnetic field the axion should undergo Primakoff conversion into a real photon, with energy equal to the rest mass of the axion, and a virtual photon. This prediction has led to the construction of ADMX (Axion Dark Matter eXperiment) – intended to search for the real photon – at LLNL [35]. The detector consists of a cavity 1 m long and 0.6 m in diameter cooled to about 1.5 K in a magnetic field of 8 T (Fig. 22). The cavity has a *Q* value of about 10^5 and can be tuned over the range 0.7–0.8 GHz. The output from the cavity was originally coupled into a cooled HEMT amplifier with a noise temperature of 1.7 K; thus the system noise temperature T_s was about 3.2 K. The goal of the experiment is to look for a signal above the blackbody noise of the cavity that would signify the presence of the photon produced by the decay of an axion. Since the axion energy is unknown, one must sweep the frequency of the cavity.

The expected signal is exceedingly small. The photon power generated in the cavity by the decay of the axions scales as $m_a n_a \times (\text{magnetic field})^2 \times (\text{cavity volume})$. There are two theories for the scaling coefficient, which involves the coupling strength of the axion to the two photons. The KSVZ (Kim–Shifman–Vainshtein–Zakharov) model [41, 42] yields a photon power $\delta P \approx 5 \times 10^{-22}$ W, while the DFSZ (Dine–Fischler–Srednicki–Zhitnitsky) model [43, 44], which leads to a weaker coupling, predicts $\delta P \approx 5 \times 10^{-23}$ W. To achieve the DFSZ limit with a signal–to–noise ratio (SNR) of 4, one can show that the frequency scan rate is given by

$$df/dt \approx (80\ \mathrm{MHz/yr})(f/1\ \mathrm{GHz})^2. \qquad (6.1)$$

Equation (6.1) can be rewritten as $df/f^2 \approx 2.5 \times 10^{-18}\, dt$, where f is in hertz and t is in seconds. This result, in turn, can be integrated to find the time $\tau(f_1, f_2)$ to scan from a lower frequency f_1 to an upper frequency f_2:

$$\tau(f_1, f_2) \approx 4 \times 10^{17}(1/f_1 - 1/f_2)\ \mathrm{s}. \qquad (6.2)$$

For the frequency decade $f_1 = 0.24$ GHz to $f_2 = 0.48$ GHz, one finds a scan time of about 270 years.

Fortunately, there is every reason to believe that this unrealistically long scan time can be drastically reduced. For a power detector with noise temperature T_s and bandwidth Δf, the Dicke radiometer equation [45] yields an integration time $\tau(f_1,f_2)$ given by

$$\mathrm{SNR} = (\delta P/k_B T_s)\,[\,\tau(f_1,f_2)\,/\Delta f]^{1/2}. \quad (6.3)$$

Thus, for given values of SNR, δP and Δf, we see that $\tau(f_1,f_2) \propto T_s^2$. If one were to cool the cavity with a dilution refrigerator to (say) 50 mK while retaining the existing amplifier, the value of T_s would be reduced by a factor of about 2 and hence $\tau(f_1,f_2)$ would be shortened by a factor of about 4. However, if instead one were to cool the cavity to 50 mK and replace the HEMT amplifier with a MSA, also cooled to 50 mK to produce a noise temperature of 50 mK, the system noise temperature would be reduced to 100 mK. Consequently, the scan time would be reduced from 270 years by a factor of $(3.2/0.1)^2$ to about 8 months! Thus, the potential impact of the microstrip SQUID on this important cosmological experiment is extraordinary, and would enable one to test the DFSZ limit over a decade of frequency in a very accessible time.

The very low noise temperature of the MSA spurred a first, proof-of principle upgrade of the axion detector in which the HEMT was replaced with an MSA while the temperature was maintained at about 2 K. Since blackbody noise from the cavity was not reduced, the decrease in scan time was modest. Rather, the object of the upgrade was to demonstrate that the MSA could indeed operate as expected on the axion detector. In fact, the system worked extremely well at a frequency of about 842 MHz. The run acquired 88732, 80-sec data sets, corresponding to a net 82 days of data [46].

In 2010, ADMX was moved to the University of Washington, Seattle. Here it will undergo a second upgrade, with the goal of running the entire experiment on a dilution refrigerator. This upgrade will enable a definitive search for the axion over the energy range 1 – 10 μeV.

Acknowledgments

I am grateful to the following for their enormous contributions to the work described in this chapter: Marc-Olivier André, Jost Gail, Cristoph Heiden, Claude Hilbert, Emile Hoskinson, Jed Johnson, Darin Kinion, Roger Koch, Chris Macklin, John Martinis, Michael Mück, Irfan Siddiqi, Danial Slichter, Claudia Tesche, Cristian Urbina, Dale van Harlingen and Fred Wellstood.

Axion Dark Matter eXperiment (ADMX)

References

[1] London, F. (1950) *Superfluids*, Wiley, New York .

[2] Josephson, B. D. (1962) Possible new effects in superconductive tunneling, *Phys. Lett.* **1**, 251-253; Supercurrents through barriers, (1965) *Adv. Phys.* **14**, 419-451.

[3] Jaklevic, R. C., Lambe, J., Silver, A. H., and Mercereau, J. E. (1964) Quantum interference effects in Josephson tunneling, *Phys. Rev Lett.* **12**, 159-160.

[4] Zimmerman, J. E., Thiene, P., and Harding. J. T. (1970) Design and operation of stable rf-biased superconducting point-contact quantum devices, and a note on the properties of perfectly clean metal contacts, *J. Appl. Phys*. **41**, 1572-1580.

[5] Mercereau, J. E. (1970) Superconducting magnetometers, *Rev. Phys. Appl.* **5**. 13-20; Nisenoff. M. (1970) Superconducting magnetometers with sensitivities approaching 10^{-10} gauss, *Rev. Phys. Appl.* **5**, 21-24.

[6] D. Koelle, R. Kleiner, F. Ludwig, E. Dantsker and John Clarke, High-Transition-Temperature Superconducting QUantum Interference Devices, *Rev. Mod. Phys.* **71**, 631-686 (1999).

[7] The SQUID Handbook Vol. I *Fundamentals and Technology of SQUIDs and SQUID Systems*, (eds. John Clarke, Alex I. Braginski) Wiley-VCH, Weinheim, Germany (2004).

[8] The SQUID Handbook Vol. II *Applications of SQUIDs and SQUID Systems*, (eds. John Clarke, Alex I. Braginski) Wiley-VCH, Weinheim, Germany (2006).

[9] Stewart, W. C. (1968) Current-voltage characteristics of Josephson junctions, *Appl. Phys. Lett.* **12**, 277-280.

[10] McCumber, D. E. (1968) Effect of ac impedance on dc voltage-current characteristics of Josephson junctions, *J. Appl. Phys*. **39**, 3113-3118.

[11] Ambegaokar, V. and Halperin, B. I. (1969) Voltage due to thermal noise in the dc Josephson effect, *Phys. Rev. Lett.* **22**, 1364-1366.

[12] Likharev, K. K. and Semenov, V. K. (1972) Fluctuation spectrum in superconducting point junctions, *Pis'ma Zh. Eksp. Teor. Fiz.* **15**, 625-629. [(1972) *JETP Lett.* **15**, 442-445].

[13] Vystavkin, A. N., Gubankov, V. N., Kuzmin, L. S., Likharev, K. K., Migulin, V. V., and Semenov, V. K. (1974) S-c-s junctions as nonlinear elements of microwave receiving devices, *Phys. Rev. Appl.* **9**, 79-109.

[14] Koch, R. H., Van Harlingen, D. J., and Clarke, J. (1980) Quantum noise theory for the resistively shunted Josephson junction, *Phys. Rev Lett.* **45**, 2132-2135.

[15] R.H. Koch, D.J. Van Harlingen and J. Clarke, Observation of Zero-Point Fluctuations on a Resistively-Shunted Josephson Tunnel Junction, *Phys. Rev. Lett.* **47**, 1216 (1981).

[16] Tesche, C. D. and Clarke. J. (1977) dc SQUID: noise and optimization *J. Low. Temp. Phys*. **27**, 301-331.

[17] Tesche, C. D. and Clarke, J. (1979) DC SQUID: current noise, *J. Low Temp. Phys*. **37**, 397-403.

[18] Ketchen, M. B., and Jaycox, J. M. (1982) Ultra-low noise tunnel junction dc SQUID with a tightly coupled planar input coil, *Appl. Phys. Lett*. **40**, 736-738.

[19] J. M. Rowell, M. Gurvitch and J. Geerk, TITLE *Phys. Rev. B* **24**, 2278 (1981).

[20] J.M. Martinis and J. Clarke, Signal and Noise Theory for a dc SQUID Amplifier, *J. Low Temp. Phys.* **61**, 227 (1985)

[21] C. Hilbert and J. Clarke, Measurements of the Dynamic Input Impedance of a dc SQUID, *J. Low Temp. Phys.* **61**, 237 (1985).

[22] C. Hilbert and J. Clarke, DC SQUIDs as Radiofrequency Amplifiers, C. Hilbert and J. Clarke, *J. Low Temp. Phys.* **61**, 263 (1985).

[23] C. Caves, Quantum limits on noise in linear amplifiers, *Phys. Rev.* D **26**, 1817 (1982).

[24] Koch, R. H., Van Harlingen, D. J., and Clarke, J. (1980) Quantum noise theory for the dc SQUID, *Appl. Phys. Lett*. **38**, 380 (1981).

[25] M. Mück, M-O André, J. Clarke, J. Gail, Ch. Heiden, Radio frequency amplifier based on a niobium dc superconducting quantum interference device with microstrip input coupling, *Appl. Phys. Lett.* **72**, 2885 (1998).
[26] S. Ramo, J. R. Whinnery, and T. van Duzer, *Fields and Waves in Communication Electronics* (J. Wiley & Sons, New York, 1965).
[27] M. Mück and J. Clarke, The superconducting quantum interference device microstrip amplifier: computer models, *J. Appl. Phys.* **88**, 6910 (2000).
[28] M. Mück, M-O André, J. Clarke, J. Gail and Ch. Heiden, The microstrip superconducting quantum interference device rf amplifier: tuning and cascading, *Appl. Phys. Lett.* **75**, 3545 (1999).
[29] R. E. Collin, *Foundations for Microwave Engineering*, 2nd ed. (IEEE, New York, 2001).
[30] D. Kinion and John Clarke, Microstrip SQUID Radio-Frequency Amplifier: Scattering Parameters and Input Coupling, *Appl. Phys. Lett.* **92** 172503 (2008).
[31] D. Kinion and John Clarke, Near-Quantum-Limited Amplifier for the Axion Dark-Matter Experiment, *Appl.Phys. Lett.* **98**, 202503 (2011).
[32] F. C. Wellstood, C. Urbina and John Clarke, *Hot Electron Effects in Metals,* Phys. Rev. B, **49**, 5942 (1994).
[33] L. Spietz, K. Irwin, and J. Aumentado, Appl. Phys. Lett. 95, 092505 (2009).
[34] D. Hover, Y-F Chen, G. Bibeill, L. Maurer, S. Sendelbach, and R. McDermott, APS March Meeting 2011.
[35] R. Bradley, J. Clarke, D. Kinion, L.J. Rosenberg, K. van Bibber, S. Matsuki, M. Mück, P. Sikivie, Microwave Cavity Searches for Dark–Matter Axions, *Rev. Mod. Phys.* **75**, 777 (2003).
[36] R. Peccei and H. Quinn, CP conservation in the presence of pseudoparticles, *Phys. Rev. Lett.* **38**, 1440 (1977).
[37] S. Weinberg, A new light boson?, *Phys. Rev. Lett*. **40**, 223 (1978).
[38] F. Wilczek, Problem of strong P– and T–invariance in the presence of instantons, *Phys. Rev. Lett.* **40**, 279 (1978).
[39] P. Sikivie, Experimental tests of the ‘invisible’ axion, *Phys. Rev. Lett*. **51**, 1415 (1983).
[40] P. Sikivie, Detection rates for ‘invisible’–axion searches, *Phys. Rev. D* **32**, 2988 (1985).
[41] J.E. Kim, Weak–interaction singlet and strong CP invariance, *Phys. Rev. Lett.* **43**, 103 (1979).
[42] M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Can confinement ensure natural CP invariance of strong interactions?, *Nucl. Phys. B* **166**, 493 (1980).
[43] M. Dine, W. Fischler, M. Srednicki, A simple solution to the strong CP problem with a harmless axion, *Phys. Lett. B* **104**, 199 (1981).
[44] A.P. Zhitnitsky, On the possible suppression of axion–hadron interactions, *Yad. Fiz.* **31**, 497 (1980) [*Sov. J. Nucl. Phys*. **31**, 260 (1980)].
[45] R.H. Dicke, The measurement of thermal radiation at microwave frequencies, *Rev. Sci. Instrum.* **17**, 268 (1946).
[46] S.J. Asztalos *et al., Phys. Rev. Lett*. **104**, 041301(2010).