Measurement of transverse Emittance

The emittance characterizes the whole beam quality, assuming linear behavior as described by second order differential equation.

It is defined within the phase space as: $\varepsilon_x = \frac{1}{\pi}\int_A dx dx'$

Or using the density function *ρ(x,x')* via: $\varepsilon_x = \frac{1}{\pi}\int \rho(x,x')\, dx dx'$

The measurement is based on determination of:

either profile width σ_x and angular width σ_x' at one location
or σ_x at different locations and linear transformations.

Different devices are used at transfer lines:

- Lower energies E_{kin} < 100 MeV/u: slit-grid device, pepper-pot
  (suited in case of non-linear forces).
- All beams: Quadrupole variation, 'three grid' method using linear transformations
  (**not** well suited in the presence of non-linear forces)

Synchrotron: lattice functions results in stability criterion

⇒ beam width delivers emittance: $\varepsilon_x = \frac{1}{\beta_x(s)}\left[\sigma_x^2 - \left(D(s)\frac{\Delta p}{p}\right)^2\right]$ and $\varepsilon_y = \frac{\sigma_y^2}{\beta_y(s)}$

Basic Equations for transverse Emittance

Beam matrix at one location: $\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} = \varepsilon \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$ with $\vec{x} = \begin{pmatrix} x \\ x' \end{pmatrix}$

The value of emittance is:

$$\varepsilon_x = \sqrt{\det \boldsymbol{\sigma}} = \sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}$$

For the profile and angular measurement:

$$x_\sigma = \sqrt{\sigma_{11}} = \sqrt{\varepsilon\beta} \quad \text{and}$$

$$x'_\sigma = \sqrt{\sigma_{22}} = \sqrt{\varepsilon\gamma}$$

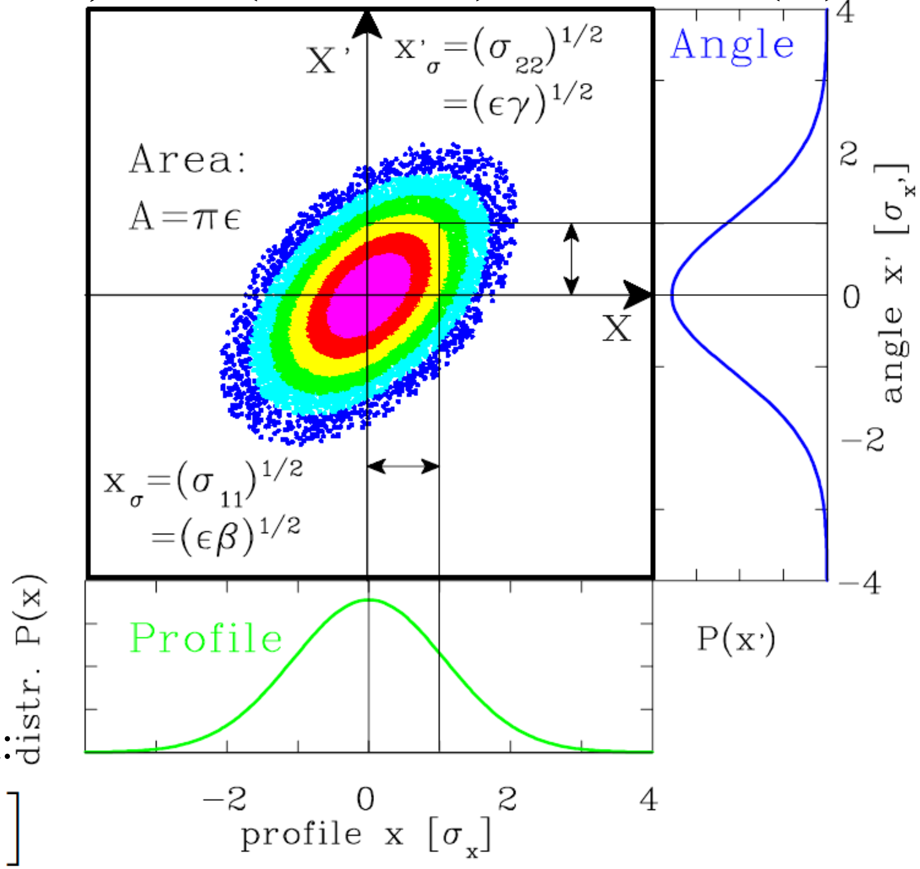
Using the Twiss Parameters:

$$\varepsilon_x = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

The density function for a Gaussian distribution:

$$\rho(x, x') = \frac{1}{2\pi\epsilon} \exp\left[-\frac{1}{2}\, \vec{x}^T\, \sigma^{-1}\, \vec{x}\,\right]$$

$$= \frac{1}{2\pi\epsilon} \exp\left[\frac{-1}{2\det\sigma}\left(\sigma_{22}x^2 - 2\sigma_{12}xx' + \sigma_{11}x'^2\right)\right]$$



The Emittance for non-Gaussian Beams

The beam distribution can be non-Gaussian.

Examples: ➢ proton beams behind ion source

➢ space charged dominated beams at LINAC and synchrotron

➢ cooled beams at storage rings.

General description of emittance: $\varepsilon_{rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$

The n^{th} central moment of a density distribution $\rho(x, x')$ calculated via

$$\mu \equiv \langle x \rangle = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot \rho(x, x') dx' dx}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x, x') dx dx'} \quad \text{and} \quad \langle x^n \rangle = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu)^n \cdot \rho(x, x') dx' dx}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x, x') dx dx'}$$

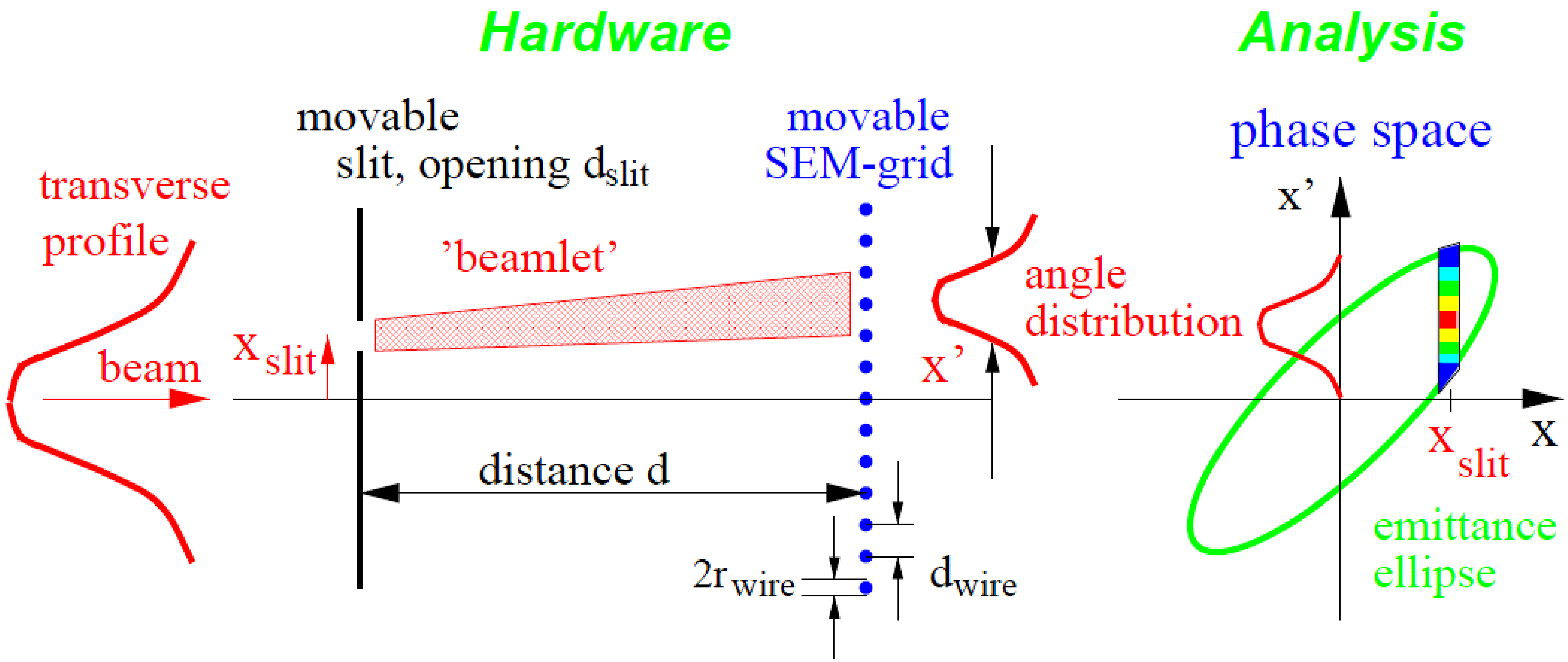
$\epsilon_{rms} \leftrightarrow$ beam fraction for *Gaussian* distribution: $\epsilon(f) = -2\pi\epsilon_{rms} \cdot \ln(1 - f)$

factor to ϵ_{rms}	$1 \cdot \epsilon_{rms}$	$\pi \cdot \epsilon_{rms}$	$2\pi \cdot \epsilon_{rms}$	$4\pi \cdot \epsilon_{rms}$	$6\pi \cdot \epsilon_{rms}$	$8\pi \cdot \epsilon_{rms}$
faction of beam f [%]	15	39	63	86	95	98

Care: No common definition for value of emittance.

Slit-Grid: Direct determination of position and angle distribution.

Used for protons/heavy ions with E_{kin} < 100 MeV/u ⇔ range R < 1 cm.



Slit: position ***P(x)*** with typical width: 0.1 to 0.5 mm

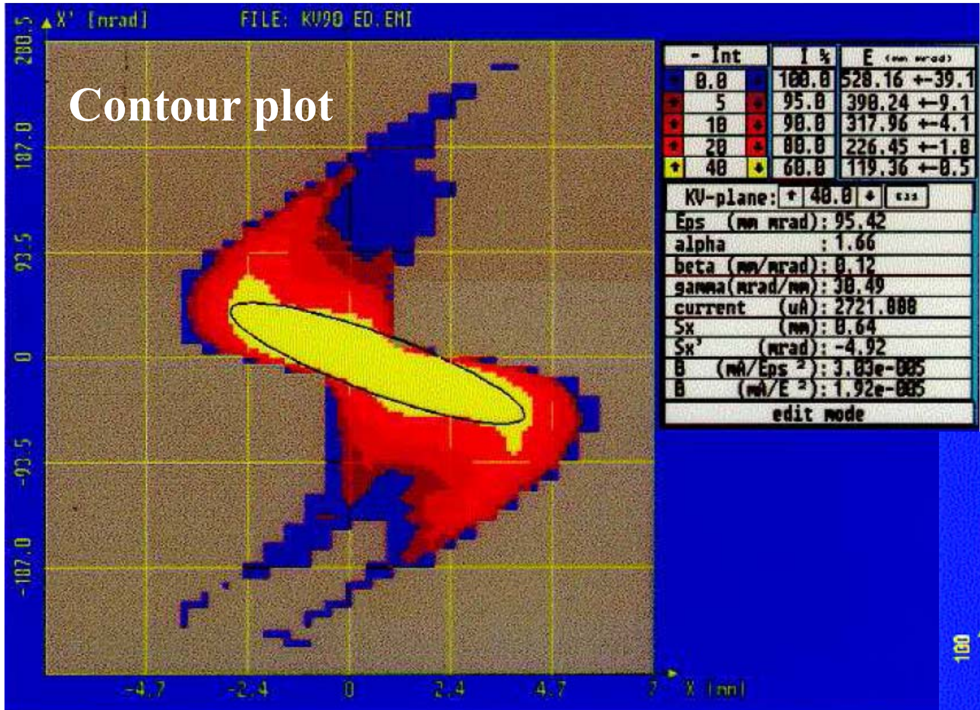
Distance: 10 cm to 1 m (depending on beam velocity)

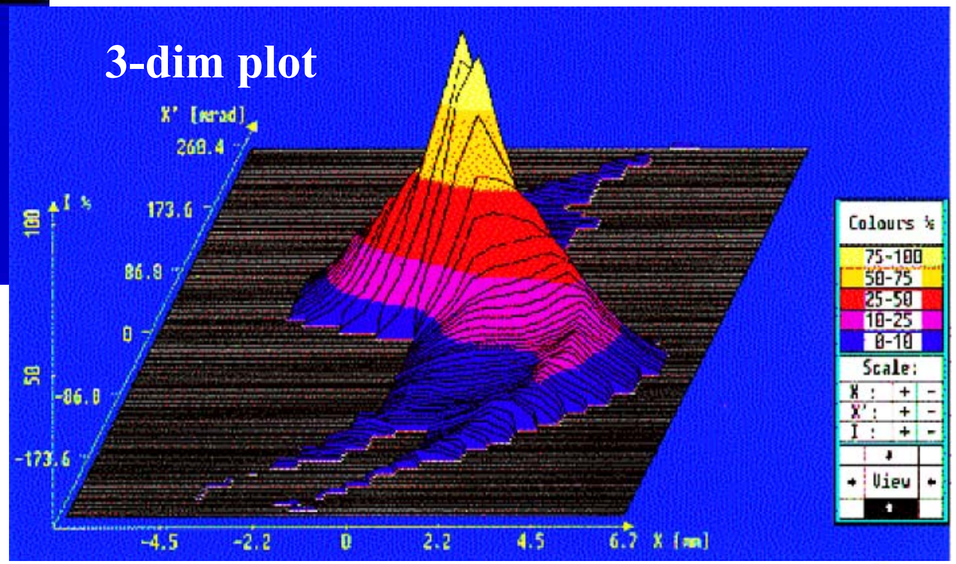
SEM-Grid: angle distribution ***P(x')***

Result for a beam behind ion source: ➢ here aberration in quadrupoles due to large beam size

➢ different evaluation and plots possible

➢ can monitor any 1-dim distribution





Low energy ion beam:
⇒ well suited for emittance showing space-charge effects or aberrations.

The Resolution of a Slit-Grid Device

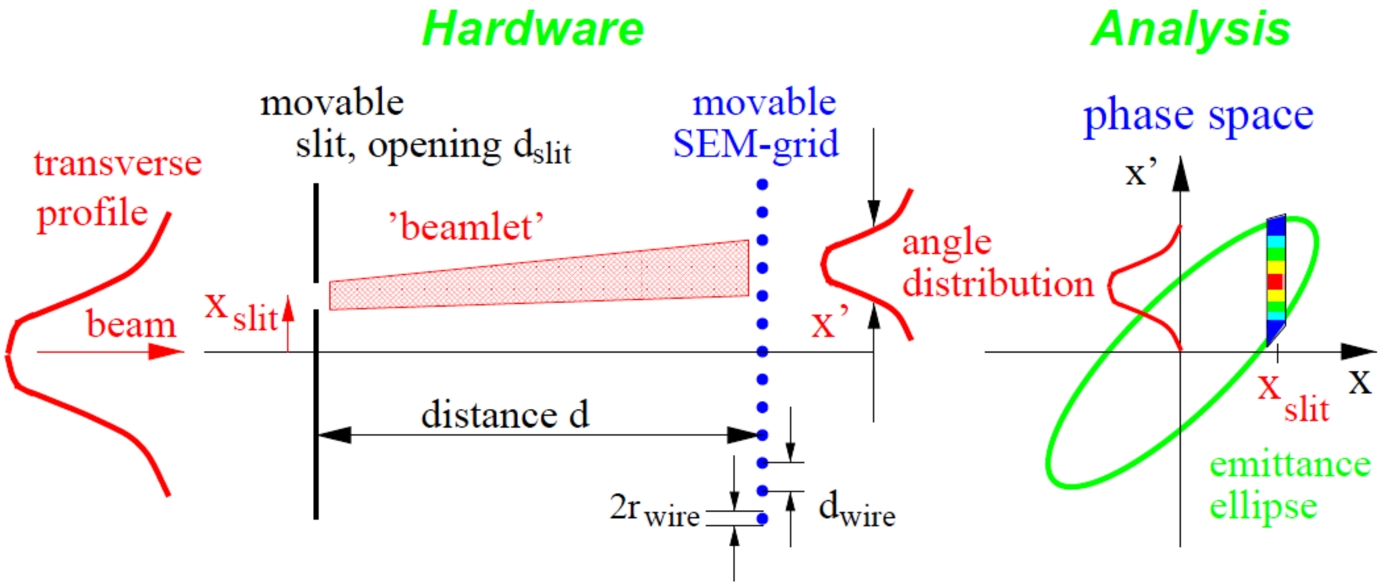
The width of the slit d_{slit} gives the resolution in space $\Delta x = d_{slit}$.

The angle resolution is $\Delta x' = (d_{slit} + 2r_{wire})/d$

$\Rightarrow$ discretization element $\Delta x \cdot \Delta x'$.

By scanning the SEM-grid the angle resolution can be improved.

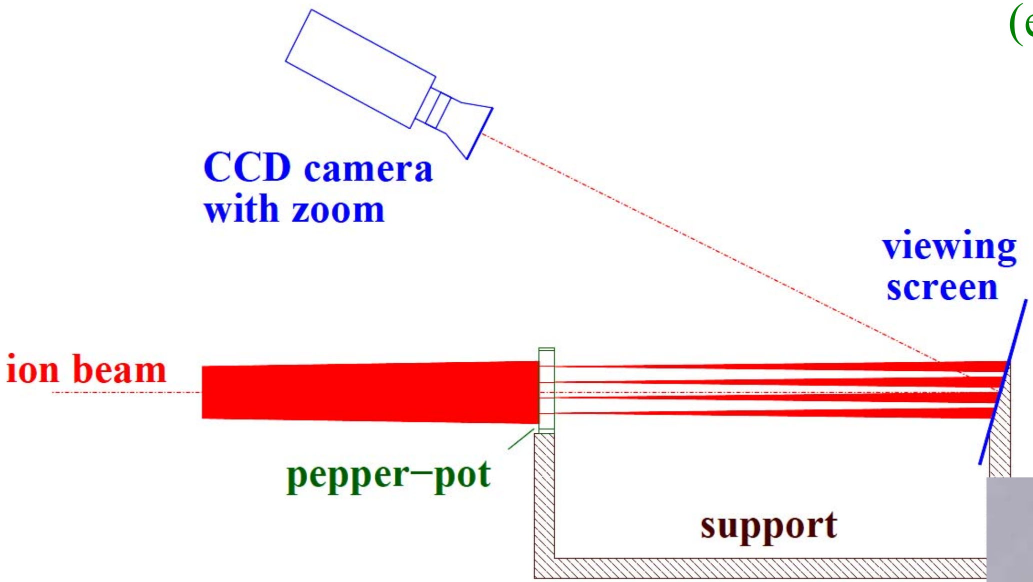
Problems for small beam sizes or parallel beams.



For pulsed LINACs: Only one measurement each pulse → long measuring time required.

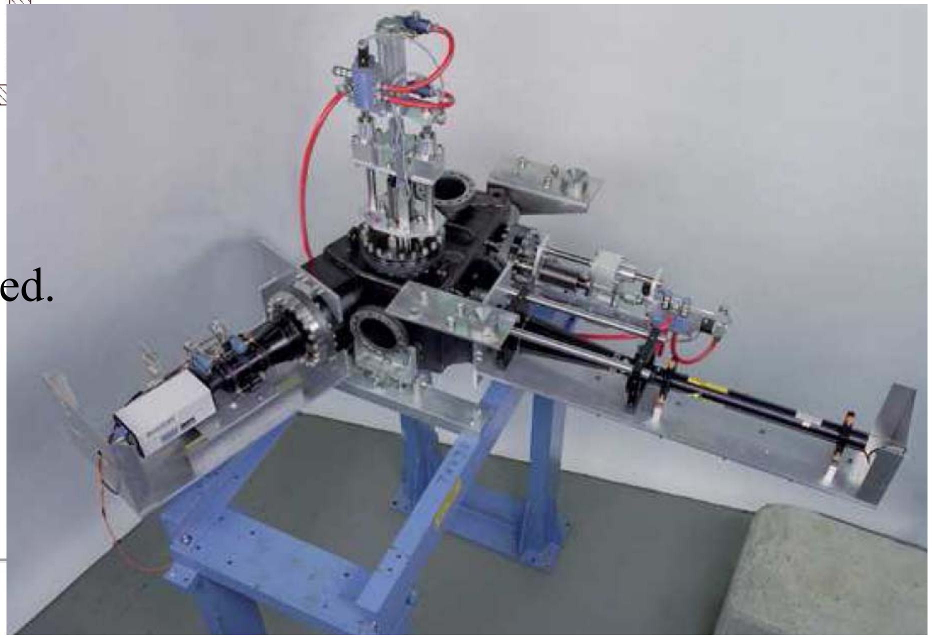
The Pepperpot Emittance Device

- For pulsed LINAC: Measurement within one pulse is an advantage
- If horizontal and vertical direction coupled → 2-dim evaluation **required** (e.g. for ECR ion source)



Example GSI-LINAC:

- ***Pepper-pot***: 15 × 15 holes with Ø0.1mm on a 50 × 50 mm^2 copper plate
- ***Distance***: pepper-pot-screen: 25 cm
- ***Data acquisition***: high resolution CCD



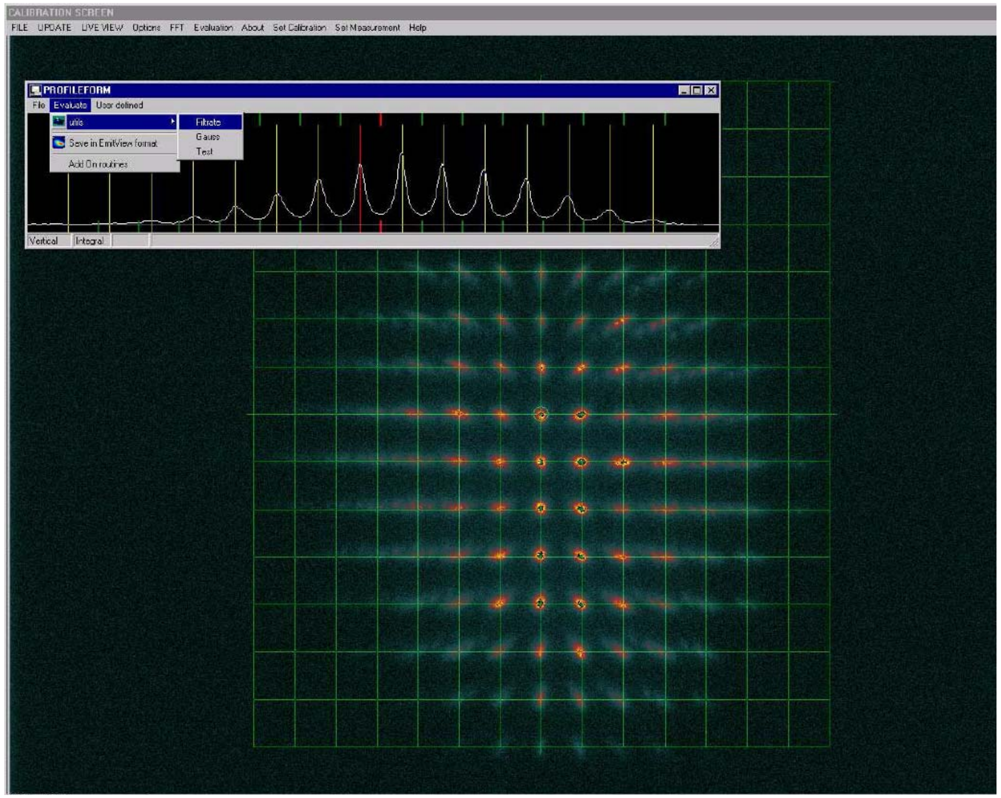
Good **spatial** resolution if many holes are illuminated.

Good **angle** resolution ***only*** if spots do not overlap.

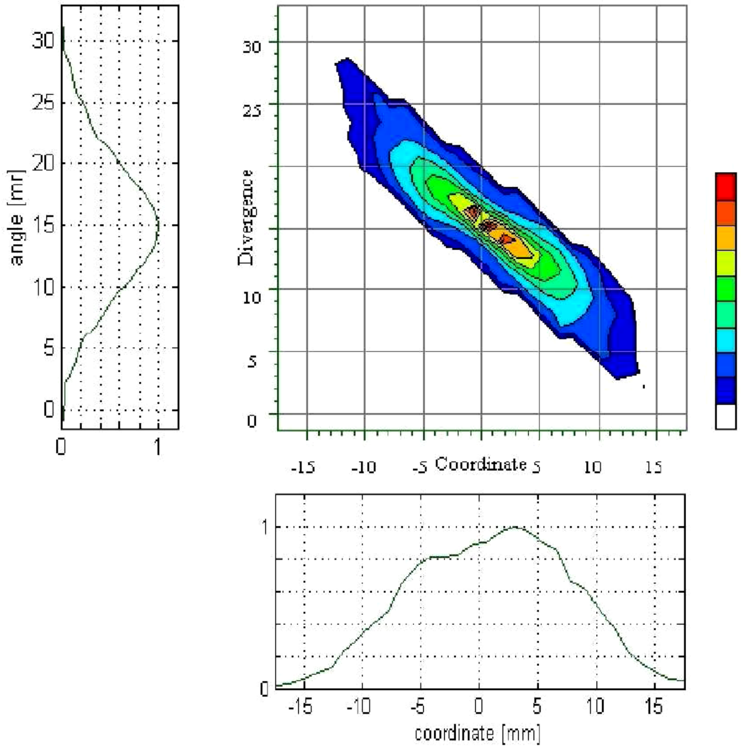
Readout by screen sometimes doubtful (!)

Result of a Pepperpot Emittance Measurement

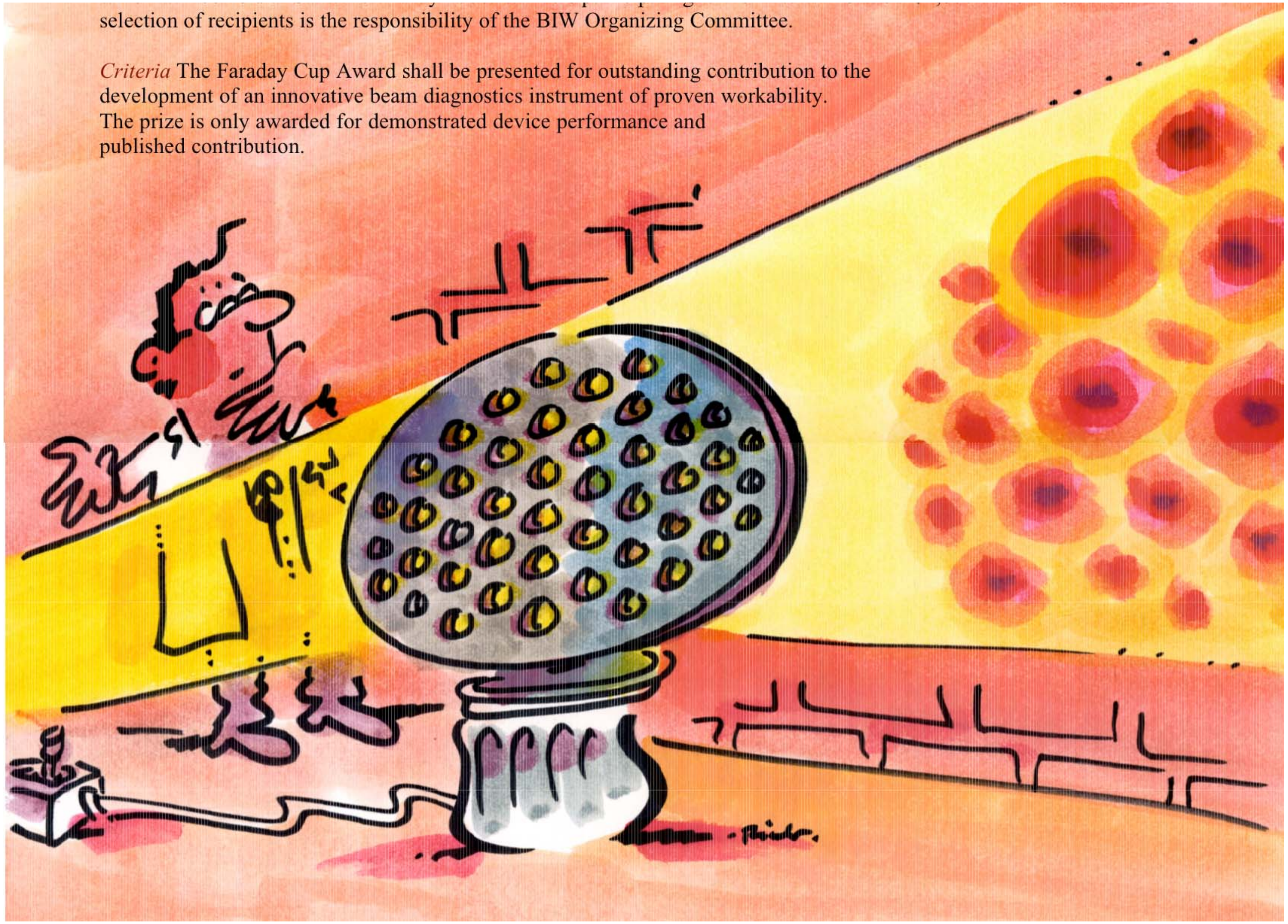
***Example*:** Ar $^{1+}$ ion beam at 1.4 MeV/u,
screen image from single shot at GSI:



Data analysis:
Projection on
horizontal and vertical plane
→ analog to slit-grid.

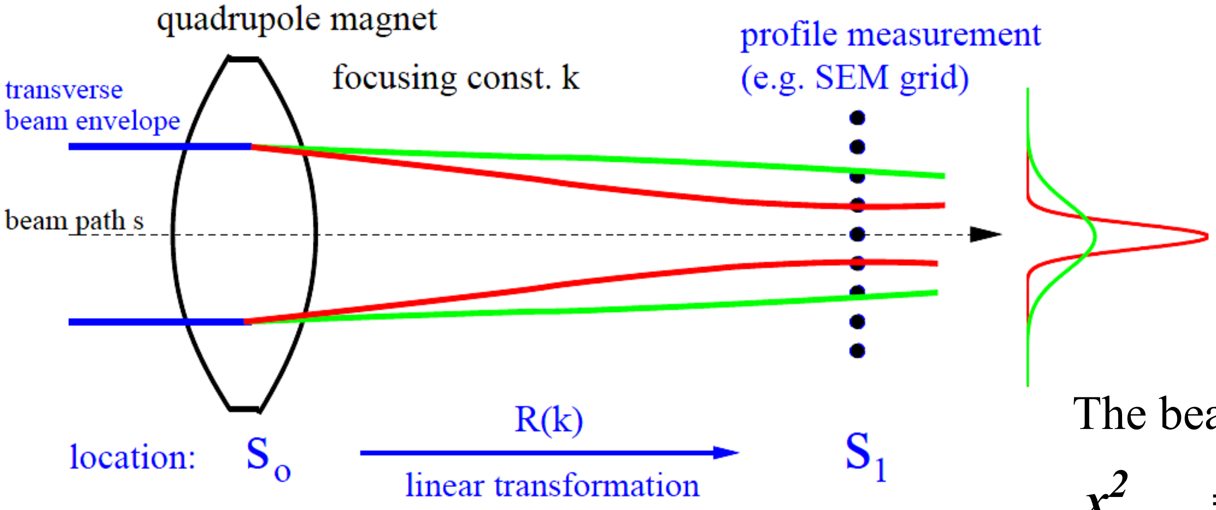


The Artist View of a Pepperpot Emittance Device



Emittance Measurement by Quadrupole Variation

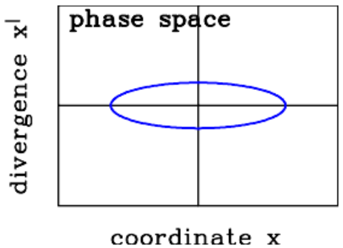
From a profile determination, the emittance can be calculated via linear transformation, if a well known and constant distribution (e.g. Gaussian) is assumed.



The beam width x_{max} and

$x^2_{max} = \sigma_{11}(1, k)$ is measured,

matrix **R(k)** describes the focusing.

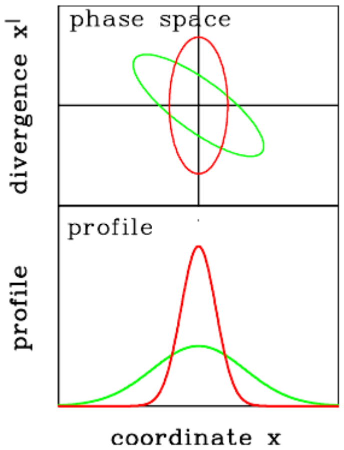


beam matrix:

(Twiss parameters)

$\sigma_{11}(0)$, $\sigma_{12}(0)$, $\sigma_{22}(0)$

to be determined



measurement:

$x^2(\mathbf{k}) = \sigma_{11}(1,\mathbf{k})$

Some Examples for linear Transformations

Without dispersion one can use the 2-dim sub-space (x, x').

- Drift with length L: $\mathbf{R}_{\mathbf{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$
- Horizontal *focusing* with quadrupole constant k and eff. length L:

$$\mathbf{R}_{\mathbf{focus}} = \begin{pmatrix} \cos\sqrt{k}L & \frac{1}{\sqrt{k}}\sin\sqrt{k}L \\ -\frac{1}{\sqrt{k}}\sin\sqrt{k}L & \cos\sqrt{k}L \end{pmatrix}$$

- Horizontal *de-focusing* with quadrupole constant k and eff. length L:

$$\mathbf{R}_{\mathbf{defocus}} = \begin{pmatrix} \cosh\sqrt{k}L & \frac{1}{\sqrt{k}}\sinh\sqrt{k}L \\ -\frac{1}{\sqrt{k}}\sinh\sqrt{k}L & \cosh\sqrt{k}L \end{pmatrix}$$

For a (ideal) quadrupole with field gradient $g = B_{pole}/a$, B_{pole} is the field at the pole and a the aperture, the quadrupole constant $k = |g|/(B\rho)_0$ for a magnetic rigidity $(B\rho)_0$.

Measurement of transverse Emittance

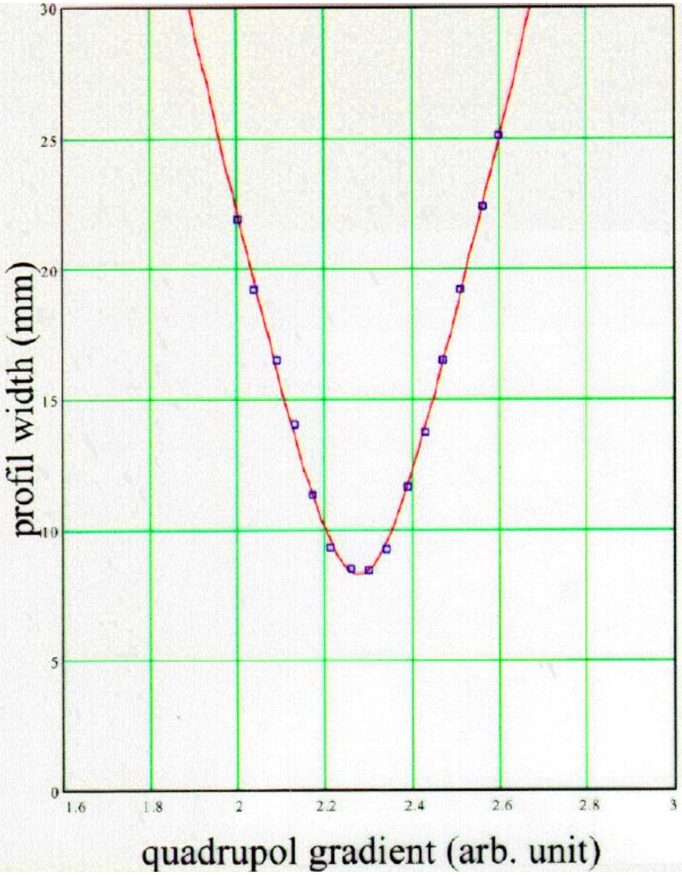
- The beam width x_{max} at s_1 is measured, and therefore $\sigma_{11}(1,k_i) = x^2_{max}(k_i)$.
- Different focusing of the quadrupole $k_1, k_2 \ldots k_n$ is used: $\Rightarrow \mathbf{R}_{\mathbf{focus}}(k_i)$, including the drift, the transfer matrix is changed $\mathbf{R}(k_i) = \mathbf{R}_{\mathbf{drift}} \cdot \mathbf{R}_{\mathbf{focus}}(k_i)$.
- Task: Calculation of *beam* matrix $\sigma(0)$ at entrance s_0 (size and orientation of ellipse)
- The transformations of the beam matrix are: $\sigma(1,k) = \mathbf{R}(k) \cdot \sigma(0) \cdot \mathbf{R}^{\mathbf{T}}(k)$.

$\Longrightarrow$ Resulting in a redundant system of linear equations for $\sigma_{ij}(0)$:

$$\begin{aligned}
\sigma_{11}(1,k_1) &= R^2_{11}(k_1)\cdot\sigma_{11}(0) + 2R_{11}(k_1)R_{12}(k_1)\cdot\sigma_{12}(0) + R^2_{12}(k_1)\cdot\sigma_{22}(0) \quad \text{focusing } k_1 \\
&\vdots \\
\sigma_{11}(1,k_n) &= R^2_{11}(k_n)\cdot\sigma_{11}(0) + 2R_{11}(k_n)R_{12}(k_n)\cdot\sigma_{12}(0) + R^2_{12}(k_n)\cdot\sigma_{22}(0) \quad \text{focusing } k_n
\end{aligned}$$

- To learn something on possible errors, $n > 3$ settings have to be performed. A setting with a focus close to the SEM-grid should be included to do a good fit.
- *Assumptions:*
  - Only elliptical shaped emittance can be obtained.
  - No broadening of the emittance e.g. due to space-charge forces.
  - If *not* valid: A self-consistent algorithm has to be used.

Example:
The beam width measured at GSI-LINAC by SEM-grid:



Simplification for 'thin lens approximation':

$$\mathbf{R}_{\mathbf{focus}}(K) = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ K & 1 \end{pmatrix}$$

$$\Rightarrow \mathbf{R}(K) = \mathbf{R}_{\mathbf{drift}} \cdot \mathbf{R}_{\mathbf{focus}} = \begin{pmatrix} 1+LK & L \\ K & 1 \end{pmatrix}.$$

Measurement of $\sigma(1,K) = \mathbf{R}(K)\sigma(0)\mathbf{R}^T(K)$

$$\begin{aligned} \sigma_{11}(1,K) &= L^2\sigma_{11}(0)\cdot K^2 \\ &\quad +2\left(L\sigma_{11}(0) + L^2\sigma_{12}(0)\right)\cdot K + L^2\sigma_{22} + \sigma_{11} \\ &\equiv aK^2 - 2abK + ab^2 + c \end{aligned}$$

The σ-matrix at quadrupole is:

$$\begin{aligned} \sigma_{11}(0) &= \frac{a}{L^2} \\ \sigma_{12}(0) &= -\frac{a}{L^2}\left(\frac{1}{L} + b\right) \\ \sigma_{22}(0) &= \frac{1}{L^2}\left(ab^2 + c + \frac{2ab}{L} + \frac{a}{L^2}\right) \end{aligned}$$

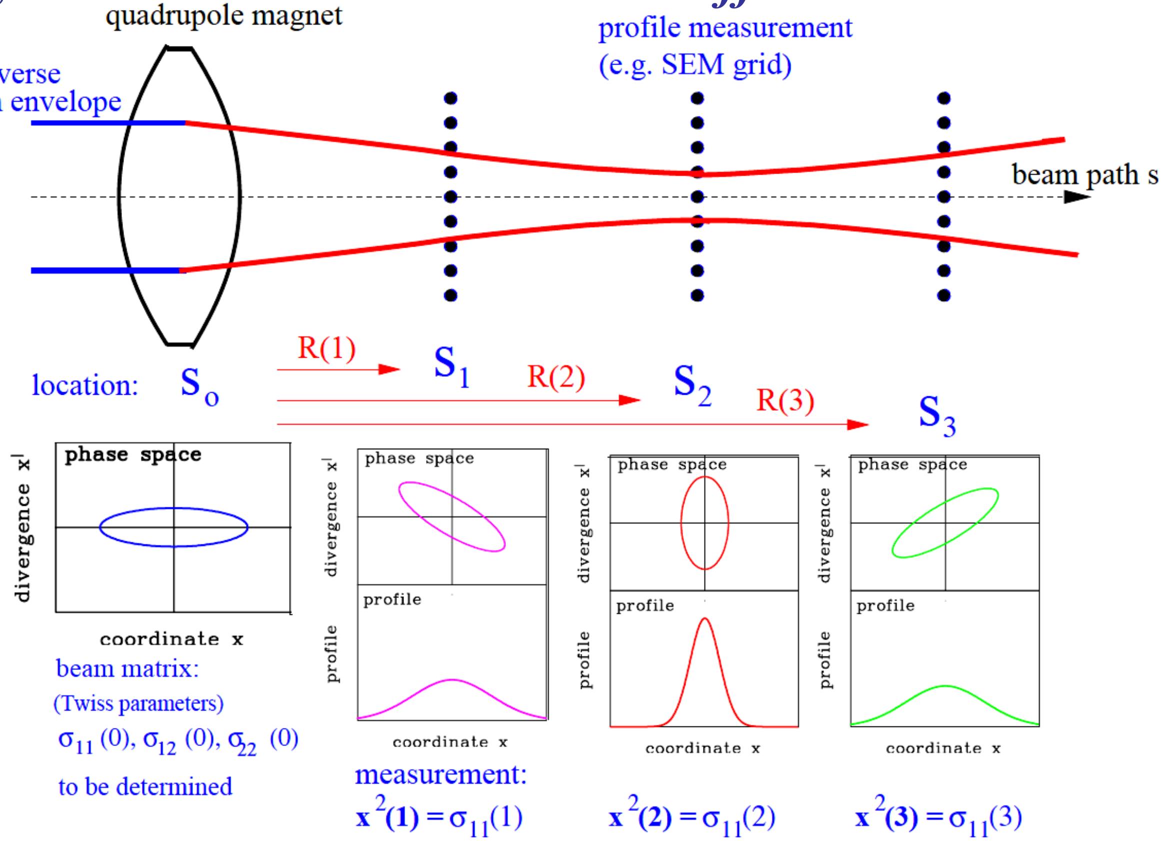
$$\epsilon = \sqrt{\det\sigma(0)} = \sqrt{\sigma_{11}(0)\sigma_{22}(0) - \sigma_{12}^2(0)} = \sqrt{ac}/L^2$$

The 'Three Grid Method' for Emittance Measurement

Instead of quadrupole variation, the beam width is measured at ***different*** locations:

The procedure is:

- Beam width $\boldsymbol{x(i)}$ measured at the locations $\boldsymbol{s_i}$
  $\Rightarrow$ beam matrix element
  $\boldsymbol{x^2(i) = \sigma_{11}(i)}$.
- The transfer matrix **R(*i*)** is known.
  (without dipole a 3 × 3 matrix.)
- The transformations are:
  $\boldsymbol{\sigma(i) = R(i)\sigma(0)R^T(i)}$
  $\Rightarrow$ redundant equations:

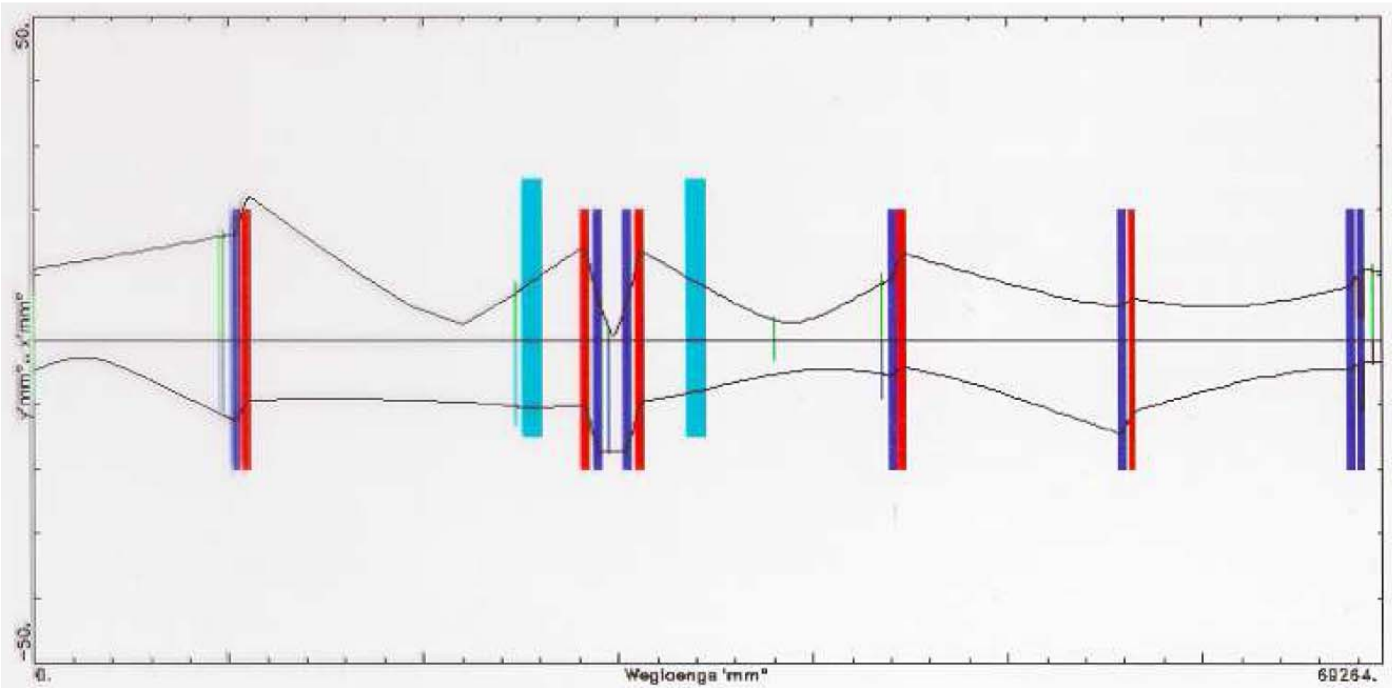


$$
\begin{aligned}
\sigma_{11}(1) &= R_{11}^2(1)\cdot\sigma_{11}(0) + 2R_{11}(1)R_{12}(1)\cdot\sigma_{12}(0) + R_{12}^2(1)\cdot\sigma_{22}(0) \qquad \mathbf{R}(1): s_0 \rightarrow s_1 \\
\sigma_{11}(2) &= R_{11}^2(2)\cdot\sigma_{11}(0) + 2R_{11}(2)R_{12}(2)\cdot\sigma_{12}(0) + R_{12}^2(2)\cdot\sigma_{22}(0) \qquad \mathbf{R}(2): s_0 \rightarrow s_2 \\
&\vdots \\
\sigma_{11}(n) &= R_{11}^2(n)\cdot\sigma_{11}(0) + 2R_{11}(n)R_{12}(n)\cdot\sigma_{12}(0) + R_{12}^2(n)\cdot\sigma_{22}(0) \qquad \mathbf{R}(n): s_0 \rightarrow s_n
\end{aligned}
$$

Results of a 'Three Grid Method' Measurement

Solution: Solving the linear equations like for quadrupole variation or fitting the profiles with linear optics code (e.g. TRANSPORT, WinAgile, MadX).

Example: The hor. and vert. beam envelope and the beam width at a transfer line:



Assumptions: ➢ constant emittance, in particular no space-charge broadening

➢100 % transmission i.e. no loss due to vacuum pipe scraping

➢ no misalignment, i.e. beam center equals center of the quadruples.

Summary for transverse Emittance Measurment

Emittance measurements are very important for comparison to theory.

It includes size (value of ε) and orientation in phase space (σ_{ij} *or* α, β and γ)

(three independent values)

Techniques for transfer lines (synchrotron: width measurement sufficient):

Low energy beams → direct measurement of x- and x'-distribution

- ***Slit-grid:*** movable slit → ***x***-profile, grid → ***x'***-profile
- ***Pepper-pot***: holes → ***x***-profile, scintillation screen → *x'*-profile

All beams → profile measurement + linear transformation:

- ***Quadrupole variation:*** one location, different setting of a quadrupole
- ***'Three grid method':*** different locations
- ***Assumptions:***
  - well aligned beam, no steering
  - no emittance blow-up due to space charge.